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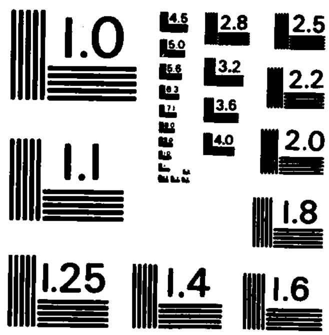

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

PRODUCT STOCHASTIC MEASURES, MULTIPLE STOCHASTIC INTEGRALS
AND THEIR EXTENSIONS TO NUCLEAR SPACE VALUED PROCESSES

by

Victor M. Perez-Abreu C.

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by

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Statistics

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VICTOR M. PEREZ-ABREU C. Product Stochastic Measures, Multiple Stochastic Integrals and their Extensions to Nuclear Space Valued Processes. (Under the direction of GOPINATH KALLIANPUR.)

A theory of L^2-valued product stochastic measures of non-identically distributed L^2-independently scattered measures is developed using concepts of symmetric tensor product Hilbert spaces. Applying the theory of vector valued measures we construct multiple stochastic integrals with respect to the product stochastic measures. A clear relationship between the theories of vector valued measures and multiple stochastic integrals is established. This work is related to the work by D. D. Engel (1982) who gives a different approach to the construction of product stochastic measures. The two approaches are compared.

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CHAPTER I

INTRODUCTION

Beginning with the works by Wiener (1938) and Itô (1951), the notion of multiple stochastic integrals (m.s.i.) has been a useful tool in applied and theoretical areas of Probability and Statistics, and several efforts have been made to build a general theory.

The purpose of the first part of this thesis (Chapters 2 and 3) is to define the symmetric tensor product measure (s.t.p.m.) and use this concept systematically to construct multiple stochastic integrals. The latter will be obtained as integrals w.r.t. appropriate product stochastic measures (p.s.m.). The concept of symmetric tensor product measures is central to the construction of product stochastic measures.

In Chapter II of this work we develop the theory of tensor and symmetric tensor products of orthogonally scattered measures. It turns out that the different powers (in the symmetric tensor product sense) of the s.t.p.m.'s are orthogonal and take values in a common, appropriately defined (exponential) Hilbert space. Then we construct multiple integrals with respect to the s.t.p. measure using the theory of integration w.r.t. vector valued measures as developed, for example, in the book by Dunford and Schwartz (1958).

In Chapter III we apply the results of the previous chapter to study symmetric tensor product stochastic measures and multiple stochastic integrals of dependent, non-identically distributed L^2-valued independently scattered measures. We identify the appropriate exponential Hilbert space

which is the common range for the s.t.p. stochastic measures and m.s.i. of different orders. We investigate the Gaussian and Poisson situations separately since a more general treatment is possible for these cases. We show that the symmetric tensor product measure approach includes the usual multiple Wiener and Poisson integrals defined by Itô (1951) and Ogura (1972) respectively, as well as the multiple Wiener integrals with dependent integrators of Fox and Taqqu (1984). This establishes a clear relationship between the theory of multiple stochastic integrals and the theory of vector valued measures. We conclude Chapter III with a comparison of our results with previous attempts to define product stochastic measures including the one by Engel (1982) on the L^2-theory of products of independently scattered measures.

The second part of this work (Chapters 3 and 4) deals with stochastic processes taking values in infinite dimensional spaces. Realistic models for the investigation of many important problems in Physics, Statistical Mechanics, Geophysics and certain areas of Biology, lead to stochastic processes of this kind. A convenient choice of infinite dimensional spaces is the class of nuclear spaces (more precisely, duals of nuclear spaces).

Nuclear space valued stochastic processes have been considered in the works of K. Itô (1978a, 1978b, 1983) and the papers, among others, of Dawson and Salehi (1980), Shiga and Shimizu (1980), Mitoma (1981a, 1981b) and Miyahara (1981). In most of the above papers, the nuclear space considered is $\mathcal{S}(\mathbb{R}^d)$, whose dual $\mathcal{S}(\mathbb{R}^d)'$ is the space of tempered distributions. However, in several practical problems, e.g. those occurring in neurophysiology, it is not possible to fix in advance the space in which the stochastic processes take their values (see Kallianpur and Wolpert (1984)).

We begin Chapter IV by defining the nuclear space Φ we are going to

consider in the remainder of this work. Then we define Φ'-valued Wiener processes with continuous positive definite bilinear form Q on $\Phi \times \Phi$, and study some related concepts that are used later in this work. We conclude Chapter IV by presenting stochastic integrals with respect to a Φ'-valued Wiener process.

The main object of the second part of this work is to develop techniques for the study of nonlinear functionals of W_t. In this direction we construct in Chapter V multiple Wiener integrals (m.W.i.) with respect to a Φ'-valued Wiener process W_t. This construction leads to real valued, finite dimensional multiple stochastic integrals w.r.t. dependent non-identically distributed independently scattered measures of the kind considered in the first part of this work. In addition we obtain the Wiener decomposition of the space of Φ'-valued nonlinear functionals of W_t as an inductive limit of appropriate Hilbert spaces. Multiple stochastic integral expansions and stochastic integral representations for nonlinear functionals of W_t are also obtained in Chapter V. It turns out that the stochastic integrals constructed in Chapter IV are the ones useful in representing nonlinear functionals and Φ'-valued square integrable martingales.

CHAPTER II

TENSOR PRODUCT AND MULTIPLE INTEGRALS OF ORTHOGONALLY SCATTERED MEASURES

We begin this chapter by presenting results on tensor products of orthogonally scattered measures. We include the infinite tensor product case (Theorem 2.1.4) which seems to be considered for the first time and appears as a natural generalization of Theorem 2.1.3. In Section 2.2 we obtain the symmetric tensor product measure of different orthogonally scattered measures which are mutually orthogonal over disjoint sets, and present some of its properties. Finally, in Section 2.3 we apply the theory of integration with respect to vector valued measures to define multiple integrals for the symmetric tensor product measure.

2.1 Tensor and infinite tensor product of orthogonally scattered measures.

The theory of orthogonally scattered measures with values in a Hilbert space has been presented by Masani (1968) among others. We begin this section by presenting a definition and a theorem which are given in the above named work.

Definition 2.1.1. Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, and let (T,A,μ) be a measure space where μ is a non-negative, countably additive, σ-finite measure. Let A_μ denote the ring

$$A_\mu = \{A \in A : \mu(A) < \infty\}.$$

The H-valued set function X is said to be an orthogonally scattered measure (o.s.m.) on $(T,\mathcal{A})$ with values in H and control measure μ if the next two conditions hold:

i) For each sequence $\{A_n\}_{n\geq 1}$ of disjoint elements in $\mathcal{A}_\mu$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\mu$, $\sum_{n=1}^{m} X(A_n)$ converges in H to $X(\bigcup_{n=1}^{\infty} A_n)$ when $m \to \infty$.

ii) For $A, B \in \mathcal{A}_\mu$

$$\langle X(A), X(B)\rangle_H = \mu(A \cap B). \tag{2.1.1}$$

It follows from (2.1.1) that

$$\mu(A) = \|X(A)\|_H^2 \qquad A \in \mathcal{A}_\mu \tag{2.1.2}$$

and that if $A, B \in \mathcal{A}_\mu$, $A \cap B = \phi$ then $X(A)$ and $X(B)$ are orthogonal elements in H.

The linear space of H

$$H_X = \overline{sp}\{X(A): A \in \mathcal{A}_\mu\} \tag{2.1.3}$$

is called the subspace of X. Condition (2.1.2) enables us to define a well-known isometry between H_X and $L^2(T,\mathcal{A},\mu)$ as follows:

For an $\mathcal{A}_\mu$-simple function $f(t) = \sum_{i=1}^{r} c_i 1_{A_i}(t)$, $A_i \in \mathcal{A}_\mu$, $c_i \in \mathbb{R}$ $i=1,\dots,r$ define

$$I_X(f) = \int_T f(t)dX(t) = \sum_{i=1}^{r} c_i X(A_i)$$

and for $f \in L^2(T,\mathcal{A},\mu)$ define

$$I_X(f) = \int_T f(t)dX(t) = \lim_{n\to\infty} \int_T f_n(t)dX(t)$$

where $\{f_n\}_{n\geq 1}$ is a sequence of $\mathcal{A}_\mu$-simple functions converging to f in

$L^2(T,A,\mu)$. The following result is known as the isomorphism theorem.

Theorem 2.1.1 (Masani (1968)). Let X be as in Definition 2.1.1. Then

(2.1.4) $$I_X: f \to \int_T f(t)dX(t)$$

is an isometry on $L^2(T,A,\mu)$ onto $H_X \subset H$ such that for $A \in A_\mu$, $X(A) = I_X(1_A)$. Then every o.s.m. X carries with it two Hilbert spaces H_X and $L^2(T,A,\mu)$ which are isomorphic under the isometric integral I_X.

Similar to the classical theory of real valued product measures (Halmos (1950)), it is possible to develop a theory of tensor products of orthogonally scattered measures. In this direction we give a proof of the next theorem which is established in Chevet (1981). Before doing this we first introduce some notation: For two Hilbert spaces H_1 and H_2, $H_1 \otimes H_2$ denotes their Hilbert space tensor product (Reed and Simon (1980)) with inner product $\langle\cdot,\cdot\rangle_{H_1 \otimes H_2}$, and $h_1 \otimes h_2$ denotes the tensor product of the elements $h_1 \in H_1$, $h_2 \in H_2$. Given two real valued measures μ and ν, $\mu \otimes \nu$ denotes their product measure.

Theorem 2.1.2 (Chevet (1981)). For each i=1,...,n, let (T_i,A_i,μ_i) be a σ-finite measure space, H_i a real separable Hilbert space and X_i an o.s.m. on (T_i,A_i) with values in H_i and control measure μ_i. Then there exists a unique orthogonally scattered measure $\bigotimes_{i=1}^{n} X_i$ on $(T_1\times\ldots\times T_n,\ A_1\times\ldots\times A_n)$ with values in $H_1\otimes\ldots\otimes H_n$ and control measure $\mu_1\otimes\ldots\otimes\mu_n$ such that

(2.1.5) $$\bigotimes_{i=1}^{n} X_i(A_1\times\ldots\times A_n) = X_1(A_1)\otimes\ldots\otimes X_n(A_n)$$

for $A_i \in A_{\mu_i} = \{A \in A_i:\ \mu_i(A) < \infty\}$. The o.s.m. $\bigotimes_{i=1}^{n} X_i$ is called the tensor product o.s.m. of $X_1,\ldots,X_n$.

Proof It is enough to show it for n = 2. Let

$$R = \{A_1 \times A_2 : A_i \in \mathcal{A}_i \quad i=1,2\}.$$

For $A_i \in \mathcal{A}_{\mu_i}$ $i=1,2$ define the $H_1 \otimes H_2$-valued set function

$$\bigotimes_{i=1}^{2} X_i(A_1 \times A_2) = X_1(A_1) \otimes X_2(A_2).$$

Next let $C^{(2)} = F_0(R)$ be the field generated by R, i.e. the collection of all disjoint unions of elements in R. For $C \in C^{(2)}$ with $\mu_1 \otimes \mu_2(C) < \infty$, i.e.

$$C = \bigcup_{j=1}^{r} (A_j \times B_j)$$

where $\mu_1(A_j) < \infty$, $\mu_2(B_j) < \infty$ and $A_j \times B_j$, $j=1,\ldots,r$ are disjoint elements in R, define

$$\bigotimes_{i=1}^{2} X_i(C) \overset{\text{def}}{=} \sum_{j=1}^{r} \bigotimes_{i=1}^{2} X_i(A_j \times B_j) = \sum_{j=1}^{r} X_1(A_j) \otimes X_2(B_j).$$

Thus

$$\Big|\Big| \bigotimes_{i=1}^{2} X_i(C) \Big|\Big|^2_{H_1 \otimes H_2} = \sum_{j=1}^{r} \sum_{k=1}^{r} \langle X_1(A_j), X_1(A_k)\rangle_{H_1} \langle X_2(B_j), X_2(B_k)\rangle_{H_2}.$$

But $(A_j \times B_j) \cap (A_k \times B_k) = \phi$ $j \neq k$, then

$$\langle X_1(A_j), X_1(A_k)\rangle_{H_1} \langle X_2(B_j), X_2(B_k)\rangle_{H_2} = 0 \quad j \neq k$$

and therefore

$$\Big|\Big| \bigotimes_{i=1}^{2} X_i(C) \Big|\Big|^2_{H_1 \otimes H_2} = \sum_{j=1}^{r} \mu_1(A_j)\mu_2(B_j) = \mu_1 \otimes \mu_2(C) < \infty.$$

Since $\mu_1 \otimes \mu_2$ is a σ-finite measure on $\mathcal{A}_1 \times \mathcal{A}_2$, the result follows from the Hahn extension theorem for o.s.m. given in Masani (1968).

Q.E.D.

For convenience of later reference we present the following theorem which is a special case of Theorem 2.1.2. We shall use it frequently on the remainder of the chapter. In this result we require the o.s.m. X_i's

to take values in the same Hilbert space H and the control measures μ_i to be finite.

<u>Theorem 2.1.3</u> Let X_i $i=1,\ldots,n$ be orthogonally scattered measures on $(T_i,\mathcal{A}_i)$ with values in a common Hilbert space H and corresponding finite control measures μ_i $i=1,\ldots,n$. Then there exists a unique n^{th} tensor product orthogonally scattered measure $\overset{n}{\underset{i=1}{\otimes}} X_i$ on $(T^n,\mathcal{A}^n)$ with values in $H^{\otimes n} = \overset{n}{\underset{i=1}{\otimes}} H$ and finite control measure $\overset{n}{\underset{i=1}{\otimes}} \mu_i = \mu_1\otimes\ldots\otimes\mu_n$ such that for $A_i \in \mathcal{A}_i$ $i=1,\ldots,n$

$$(2.1.6) \qquad \overset{n}{\underset{i=1}{\otimes}} X_i(A_1\times\ldots\times A_n) = X_1(A_1)\otimes\ldots\otimes X_n(A_n)$$

and

$$(2.1.7) \qquad \|\overset{n}{\underset{i=1}{\otimes}} X_i(A)\|^2_{H^{\otimes n}} = \overset{n}{\underset{i=1}{\otimes}} \mu_i(A) \qquad A \in \mathcal{A}^n$$

where $T^n = T_1\times\ldots\times T_n$ and $\mathcal{A}^n = \mathcal{A}_1\times\ldots\times\mathcal{A}_n$ is the n-fold product σ-field.

We now consider the extension of Theorem 2.1.3 to infinitely many dimensions. Suppose that $(T_i,\mathcal{A}_i,\mu_i)$ $i\geq 1$ is a sequence of probability spaces and that for each $i=1,2,\ldots$ X_i is an o.s.m. on $(T_i,\mathcal{A}_i)$ with values in a common Hilbert space H and control measure μ_i. An example of this situation is given at the end of Section 4.1.2 in Chapter 4 of this work. We use the following notation (as close as possible to Halmos (1950)):

$$T^\infty = \overset{\infty}{\underset{i=1}{\times}} T_i,\ \mathcal{A}^\infty = \overset{\infty}{\underset{i=1}{\times}} \mathcal{A}_i,\ T^{(n)} = \overset{\infty}{\underset{i=n+1}{\times}} T_i,\ T^n = \overset{n}{\underset{i=1}{\times}} T_i,\ \mathcal{A}^n = \overset{n}{\underset{i=1}{\times}} \mathcal{A}_i$$

and $$\mu^\infty = \overset{\infty}{\underset{i=1}{\otimes}} \mu_i.$$

For notation, definitions and basic results concerning infinite tensor products of Hilbert spaces, see Appendix A. Let $\underline{u} = (u_i)_{i=1}^\infty$ where $u_i = X_i(T_i)$ be the sequence of unit vectors in H from which we construct the infinite

tensor product Hilbert space $\overset{\infty\, u}{\underset{i=1}{\otimes}} H$ whose inner product is denoted by $\langle\cdot,\cdot\rangle_\infty$. For each $n \geq 1$, the n^{th} tensor product o.s.m $\overset{n}{\underset{i=1}{\otimes}} X_i$ of Theorem 2.1.3 takes values in $\overset{n}{\underset{i=1}{\otimes}} H$. This space may be seen as a subspace of $\overset{\infty\, u}{\underset{i=1}{\otimes}} H$ through the injection

$$h_1\otimes\cdots\otimes h_n \to h_1\otimes\cdots\otimes h_n \overset{\infty}{\underset{i=n+1}{\otimes}} X_i(T_i)$$

$h_i \in H \quad i=1,\ldots,n.$

Let C be the field of cylindrical sets in A^∞. For $A \in C$, $A = B \times T^{(n)}$ $B \in A^n$ some $n \geq 1$, define the $\overset{\infty\, u}{\underset{i=1}{\otimes}}$ H-valued set function X^∞ as

$$X^\infty(A) = \overset{n}{\underset{i=1}{\otimes}} X_i(B) \otimes \overset{\infty}{\underset{i=n+1}{\otimes}} X_i(T_i).$$

<u>Lemma 2.1.1</u> The $\overset{\infty\, u}{\underset{i=1}{\otimes}}$ H-valued set funtion X^∞ is well defined and finitely additive on C. Moreover,

$$\|X^\infty(A)\|_\infty^2 = \mu^\infty(A) \qquad A \in C.$$

<u>Proof</u> Suppose $A \in C$, $A = B\times T^{(n)}$, $B\in A^n$ and $A = C\times T^{(m)}$, $C\in A^m$ $m<n$. Then $B = C \times T_{m+1}\times\cdots\times T_n$ and

$$\overset{n}{\underset{i=1}{\otimes}} X_i(B) = \overset{m}{\underset{i=1}{\otimes}} X_i(C) \otimes X_{m+1}(T_{m+1})\otimes\cdots\otimes X_n(T_n) \quad .$$

Then

$$\overset{n}{\underset{i=1}{\otimes}} X_i(B) \otimes \overset{\infty}{\underset{i=n+1}{\otimes}} X_i(T_i) = \overset{m}{\underset{i=1}{\otimes}} X_i(C) \otimes \overset{\infty}{\underset{i=m+1}{\otimes}} X_i(T_i)$$

and hence X^∞ is unambiguously defined for $A \in C$.

Next, if $A_1, A_2 \in C$, $A_1\cap A_2 = \phi$, then for some $n \geq 1$, $A_1 = B_1\times T^{(n)}$, $A_2 = B_2\times T^{(n)}$ and $B_1\cap B_2 = \phi$. Hence

$$X^\infty(A_1\cup A_2) = \overset{n}{\underset{i=1}{\otimes}} X_i(B_1\cup B_2) \otimes \overset{\infty}{\underset{i=n+1}{\otimes}} X_i(T_i)$$

$$= \{ \bigotimes_{i=1}^{n} X_i(B) + \bigotimes_{i=1}^{n} X_i(B_2) \} \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i) \qquad \text{(by Theorem 2.1.3)}$$

$$= \bigotimes_{i=1}^{n} X_i(B_i) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i) + \bigotimes_{i=1}^{n} X_i(B_2) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i) = X^{\infty}(A_1) + X^{\infty}(A_2)$$

i.e., X^{∞} is additive on C.

Finally, if $A \in C$, $A = B \times T^{(n)}$, $B \in A^n$ some $n \geq 1$

$$\| X^{\infty}(A) \|_{\infty}^{2} = \| \bigotimes_{i=1}^{n} X_i(B) \|_{H^{\otimes n}}^{2} = \mu_1 \otimes \ldots \otimes \mu_n(B) = \mu^{\infty}(A).$$

Q.E.D.

In fact we now prove that X^{∞} has a σ-additive extension to A^{∞}. The following result appears to be new and due to the finiteness of μ_{∞} its proof does not need the Hahn extension theorem for o.s.m. of Masani (1968).

Theorem 2.1.4 There exists a unique $\overset{\infty u}{\underset{i=1}{\otimes}}$ H-valued orthogonally scattered measure X^{∞} on (T^{∞}, A^{∞}) with control measure μ^{∞} (a probability measure), such that for every $A \in C$ $A = B \times T^{(n)}$, $B \in A^n$

$$X^{\infty}(A) = \bigotimes_{i=1}^{n} X_i(B) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i) \tag{2.1.8}$$

and for every $A \in A^{\infty}$

$$\| X^{\infty}(A) \|_{\infty}^{2} = \mu^{\infty}(A). \tag{2.1.9}$$

Proof Step 1: X^{∞} is σ-additive on C.

Let $A_i \in C$ $i=1,2,\ldots$ $A_i \downarrow \phi$ (null set). We need to show that $\| X^{\infty}(A_i) \|_{\infty}^{2} \to 0$ as $i \to \infty$. But this follows since from Lemma 2.1.1 $\| X^{\infty}(A_i) \|_{\infty}^{2} = \mu^{\infty}(A_i)$ and $\mu^{\infty}(A_i) \to 0$ as $i \to \infty$ for μ^{∞} is a probability measure on A^{∞}.

Step 2: Extension to A^{∞}.

Let $A \in A^{\infty} = \sigma(C)$, then there exists a sequence of sets $\{A_n\}_{n \geq 1}$ in C

such that $\mu^{\infty}(A\Delta A_n) \to 0$ as $n\to\infty$ and $\{X^{\infty}(A_n)\}_{n\geq 1}$ is a Cauchy sequence in $\overset{\infty}{\underset{i=1}{\otimes}}{}^{\underline{u}} H$:

$$\| X^{\infty}(A_n) - X^{\infty}(A_m) \|_{\infty}^{2} = \mu^{\infty}(A_n \Delta A_m) \to 0 \quad n,m \to \infty \ .$$

Then define $X^{\infty}(A) = \|\cdot\|_{\infty}\text{-}\lim X^{\infty}(A_n)$ which has the required properties.

Q.E.D.

Corollary 2.1.1 Let $E_i \in \mathcal{A}_i$ $i \geq 1$, $A = \overset{\infty}{\underset{i=1}{\times}} E_i$. Then

$$X^{\infty}(A) = \overset{\infty}{\underset{i=1}{\otimes}} X_i(E_i) \ .$$

Proof It is known (Halmos (1950)) that $A \in \mathcal{A}^{\infty}$ and

$$\mu^{\infty}(A) = \prod_{i=1}^{\infty} \mu_i(E_i).$$

This last expression implies that

$$\sum_{i=1}^{\infty} |(\mu_i(E_i))^{\frac{1}{2}} - 1| < \infty$$

and

$$\sum_{i=1}^{\infty} |\mu_i(E_i) - 1| < \infty \quad .$$

Then conditions (1) and (2) of Proposition A.1 in Appendix A are satisfied and hence $\overset{\infty}{\underset{i=1}{\otimes}}{}^{\underline{u}} X_i(E_i)$ is a decomposable vector. The result follows by taking $A_n = E_1 \times \ldots \times E_n \times T^{(n)}$ $n \geq 1$ in the final part of the proof of the last theorem.

Q.E.D.

Corollary 2.1.2

$$X^{\infty}(T^{\infty}) = \overset{\infty}{\underset{i=1}{\otimes}} X_i(T_i)$$

and

$$\| X^{\infty}(T^{\infty}) \|_{\infty}^{2} = \mu^{\infty}(T^{\infty}) = 1.$$

2.2 The symmetric tensor product measure

In this section we obtain results regarding symmetric tensor products of different orthogonally scattered measures with values in the same Hilbert space. We have restricted ourselves to the case where each o.s.m. is bounded (finite control measure) and defined on a σ-field. The reason for these requirements is that we are primarily interested in using the well established theory of vector valued measures in order to construct product stochastic measures and their corresponding integrals, and in this theory these requirements are needed. This limitation may not be very restrictive since one could always study the limit behavior of the product measure or the integral when the control measure goes to infinite, as indeed we do in Chapter 5. Moreover, bounded orthogonally scattered measures have recently become of interest (see Niemi (1984)).

Assumption 2.2.1 Throughout this section, unless otherwise stated, we will make the following assumptions: Let $(T,\mathcal{A})$ be an arbitrary measurable space, H a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, and n a fixed but arbitrary natural number. Let X_i $i=1,\dots,n$ be o.s.m.'s on $\mathcal{A}$, taking values in H and with corresponding finite control measures μ_i $i=1,\dots,n$. Assume there exist real valued set functions μ_{ij} such that for $A,B \in \mathcal{A}$

(2.2.1) $$\mu_{ij}(A\cap B) = \langle X_i(A), X_j(B)\rangle_H \quad i,j=1,\dots,n.$$

Applying Cauchy-Schwartz inequality it follows that for each $i,j=1,\dots,n$, μ_{ij} is a signed measure of bounded variation, where $\mu_{ii} = \mu_i$ $i=1,\dots,n$. At times we shall use the facts that if μ_o is a σ-finite non-negative measure on $(T,\mathcal{A})$ such that $\mu_i \ll \mu_o$ $i=1,\dots,n$, then $\mu_{ij} \ll \mu_o$ $i,j=1,\dots,n$

and if

$$(2.2.2) \qquad r_{ij}(t) = \frac{d\mu_{ij}}{d\mu_o}(t)$$

then $R(t) = (r_{ij}(t))$ is an $n\times n$ non-negative definite matrix a.e. $d\mu_o$.

Symmetric tensor products In order to set up notation we now present some facts about symmetric tensor products of Hilbert spaces (see Guichardet (1972)). As in Section 2.1 let $H^{\otimes n}$ denote the n-fold tensor product of H. For $h_1\otimes\cdots\otimes h_n \in H^{\otimes n}$ $h_i \in H$ $i=1,\ldots,n$ define

$$(2.2.3) \qquad \sigma^n_{\odot}(h_1\otimes\cdots\otimes h_n) = \frac{1}{n!}\sum_{\Pi}(h_{\Pi_1}\otimes\cdots\otimes h_{\Pi_n})$$

where $\Pi = (\Pi_1,\ldots,\Pi_n)$ is a permutation of $(1,2,\ldots,n)$. The n-fold symmetric tensor product Hilbert space $H^{\odot n}$ is the closed subspace of $H^{\otimes n}$ generated by elements of the form

$$\sum_{k=1}^{m} c_k\sigma^n_{\odot}(h^k_1\otimes\cdots\otimes h^k_n)$$

where $c_k \in \mathbb{R}$, $h^k_i \in H$ $k=1,\ldots,m$, $i=1,\ldots,n$.

The operator $\sigma^n_{\odot}$ can be extended to an orthogonal projection operator on $H^{\otimes n}$ whose range is $H^{\odot n}$. We write

$$\mathop{\odot}_{i=1}^{n} h_i = h_1\odot\cdots\odot h_n$$

where

$$(2.2.4) \qquad h_1\odot\cdots\odot h_n = \sigma^n_{\odot}(h_1\otimes\cdots\otimes h_n) \quad h_i \in H \quad i=1,\ldots,n$$

and notice that for $g_i \in H$ $i=1,\ldots,n$

$$(2.2.5) \qquad \left\langle \mathop{\odot}_{i=1}^{n} h_i, \mathop{\odot}_{i=1}^{n} g_i\right\rangle_{H^{\odot n}} = \frac{1}{n!}\sum_{\Pi}\langle h_1,g_{\Pi_1}\rangle_H\cdots\langle h_n,g_{\Pi_n}\rangle_H.$$

Thus in particular, if $h^{\odot n} = \mathop{\odot}_{i=1}^{n} h$, $h \in H$, then

$$\langle h^{\circledS n}, g^{\circledS n}\rangle_{H^{\circledS n}} = (\langle f,g\rangle_H)^n \qquad g \in H \tag{2.2.6}$$

and

$$\| h^{\circledS n}\|^2_{H^{\circledS n}} = (\| h\|^2_H)^n.$$

For $h_i \in H$ $i=1,\dots,n$, $h_1 \circledS \dots \circledS h_n$ can be represented as a linear combination of elements of the form $h^{\circledS n}$ $h \in H$:

$$h_1 \circledS \dots \circledS h_n = \frac{1}{n!}\sum_{\ell=0}^{n-1}(-1)^{\ell}\sum_{N\in P_\ell}(1_{N^c}(1)h_1+\dots+1_{N^c}(n)h_n)^{\circledS n} \tag{2.2.7}$$

where P_ℓ is the set of subsets of $\{1,\dots,n\}$ with ℓ elements.

Symmetric tensor product measure The first result of this section gives the symmetric tensor product measure of $X_1,\dots,X_n$, which is a Hilbert space ($H^{\circledS n}$-valued) measure. Although it can be proved for any o.s.m.'s not necessarily bounded, we assume them as in the beginning of this section.

Theorem 2.2.1 Let $X_1,\dots,X_n$ be orthogonally scattered measures as in Assumption 2.2.1. Then there exists a unique $H^{\circledS n}$-valued measure $\underset{i=1}{\overset{n}{\circledS}} X_i$ on (T^n, A^n) such that for $A_i \in A$ $i=1,\dots,n$

$$\underset{i=1}{\overset{n}{\circledS}} X_i(A_1\times\dots\times A_n) = X_1(A_1)\circledS\dots\circledS X_n(A_n). \tag{2.2.8}$$

Proof Let $\underset{i=1}{\overset{n}{\otimes}} X_i$ be the $H^{\otimes n}$-valued o.s.m. on (T^n,A^n) given by Theorem 2.1.3. For $A \in A^n$ define

$$\underset{i=1}{\overset{n}{\circledS}} X_i(A) = \sigma^n_{\circledS}\left(\underset{i=1}{\overset{n}{\otimes}} X_i(A)\right) \tag{2.2.9}$$

where $\sigma^n_{\circledS}$ is the projection operator on $H^{\otimes n}$ with range $H^{\circledS n}$ defined in (2.2.3). Then for $A \in A^n$

$$\left\| \bigotimes_{i=1}^{n} X_i(A) \right\|_{H^{\otimes n}} \leq \left\| \bigotimes_{i=1}^{n} X_i(A) \right\|_{H^{\otimes n}} \tag{2.2.10}$$

and $\bigotimes_{i=1}^{n} X_i$ is a finitely additive $H^{\otimes n}$-valued measure on A^n. Next since $\bigotimes_{i=1}^{n} X_i$ is σ-additive on A^n - thus contiuous by above at the empty set - it follows from (2.2.10) that $\bigotimes_{i=1}^{n} X_i$ is continuous by above at the empty set and then σ-additive on A^n.

Finally, by Theorem 2.1.3, $\bigotimes_{i=1}^{n} X_i$ is the unique o.s.m. on (T^n, A^n) with values in $H^{\otimes n}$ such that

$$\bigotimes_{i=1}^{n} X_i(A_1 \times \ldots \times A_n) = X_i(A_1) \otimes \ldots \otimes X_n(A_n)$$

for $A_i \in A$, $i=1,\ldots,n$, from which (2.2.8) follows.

Q.E.D.

<u>Corollary 2.2.1</u> Under the assumptions of the last theorem, for $A \in A^n$

$$\left\| \bigotimes_{i=1}^{n} X_i(A) \right\|^2_{H^{\otimes n}} \leq \bigotimes_{i=1}^{n} \mu_i(A) \,. \tag{2.2.11}$$

The proof follows using (2.2.10) above and (2.1.7) in Theorem 2.1.3.

The above corollary gives an upper bound for the norm of $\bigotimes_{i=1}^{n} X_i(A)$, $A \in A^n$. We shall obtain an exact expression for this norm (Corollary 2.2.2) which uses the signed measures μ_{ij} defined in (2.2.1). We first present a more general result.

<u>Lemma 2.2.1</u> For $A, B \in A^n$

$$< \bigotimes_{i=1}^{n} X_i(A), \bigotimes_{i=1}^{n} X_i(B) >_{H^{\otimes n}} = \tag{2.2.12}$$

$$\frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \ldots \otimes \mu_{n\Pi_n} (A \cap B^{\Pi}) = \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \ldots \otimes \mu_{n\Pi_n} (B \cap A^{\Pi})$$

where for $A \in A^n$ and $\Pi = (\Pi_1,\ldots,\Pi_n)$ a permutation of $(1,\ldots,n)$ A^{Π} is

defined by

$$(2.2.13)\qquad A^{\Pi} = \{(t_1,\ldots,t_n) \in T^n:\ (t_{\Pi_1},\ldots,t_{\Pi_n}) \in A\}\ .$$

Proof Step 1 Assume that $A,B \in \mathcal{A}^n$, $A = A_1\times\ldots\times A_n$, $B = B_1\times\ldots\times B_n$ A_i, $B_i \in \mathcal{A}$ $i=1,\ldots,n$. Then from (2.2.1), (2.2.5) and Theorem 2.2.1

$$\begin{aligned}
<\bigotimes_{i=1}^{n} X_i(A), \bigotimes_{i=1}^{n} X_i(B)>_{H^{\otimes n}} &= <X_1(A_1)\otimes\ldots\otimes X_n(A_n), X_1(B_1)\otimes\ldots\otimes X_n(B_n)>_{H^{\otimes n}} \\
&= \frac{1}{n!}\sum_{\Pi} <X_1(A_1),\ X_{\Pi_1}(B_{\Pi_1})>_H\cdots<X_n(A_n),\ X_{\Pi_n}(B_{\Pi_n})>_H \\
&= \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}(A_1\cap B_{\Pi_1})\cdots\mu_{n\Pi_n}(A_n\cap B_{\Pi_n}) \\
&= \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}\otimes\ldots\otimes\mu_{n\Pi_n}((A_1\times\ldots\times A_n)\cap(B_1\times\ldots\times B_n)^{\Pi}) \\
&= \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}\otimes\ldots\otimes\mu_{n\Pi_n}(A\cap B^{\Pi}).
\end{aligned}$$

Step 2 Assume $A, B \in \mathcal{A}^n$ are of the form

$$A = \bigcup_{j=1}^{m_1} (A_j^1\times\ldots\times A_j^n),\quad B = \bigcup_{k=1}^{m_2} (B_k^1\times\ldots\times B_k^n)$$

where $A_j^1\times\ldots\times A_j^n$, $B_k^1\times\ldots\times B_k^n$ $j=1,\ldots,m_1$, $k=1,\ldots,m_2$ are as in Step 1 and $(A_j^1\times\ldots\times A_j^n)\cap(A_i^1\times\ldots\times A_i^n) = \phi$ $i\neq j$ and $(B_j^1\times\ldots\times B_j^n)\cap(B_k^1\times\ldots\times B_k^n) = \phi$ $j\neq k$. Then since $\bigotimes_{i=1}^{n} X_i$ is additive on $\mathcal{A}^n$, from Step 1 we have that

$$\begin{aligned}
<\bigotimes_{i=1}^{n} X_i(A), \bigotimes_{i=1}^{n} X_i(B)>_{H^{\otimes n}} &= \sum_{j=1}^{m_1}\sum_{k=1}^{m_2} <\bigotimes_{i=1}^{n} X_i(A_j^1\times\ldots\times A_j^n),\ \bigotimes_{i=1}^{n} X_i(B_k^1\times\ldots\times B_k^n)>_{H^{\otimes n}} \\
&= \frac{1}{n!}\sum_{\Pi}\sum_{j=1}^{m_1}\sum_{k=1}^{m_2} \mu_{1\Pi_1}\otimes\ldots\otimes\mu_{n\Pi_n}((A_j^1\times\ldots\times A_j^n)\cap(B_k^1\times\ldots\times B_k^n)^{\Pi}) \\
&= \frac{1}{n!}\sum_{\Pi}\sum_{j=1}^{m_1} \mu_{1\Pi_1}\otimes\ldots\otimes\mu_{n\Pi_n}((A_j^1\times\ldots\times A_j^n)\cap B^{\Pi}) \\
&= \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}\otimes\ldots\otimes\mu_{n\Pi_n}(A\cap B^{\Pi}).
\end{aligned}$$

Step 3 Assume $A, B \in A^n$. Then there exist sequences $\{A_m\}_{m\geq 1}$, $\{B_m\}_{m\geq 1}$ of sets in A^n as in Step 2 such that

$$\bigotimes_{i=1}^{n} \mu_i(A_m \Delta A) \underset{m\to\infty}{\to} 0, \quad \bigotimes_{i=1}^{n} \mu_i(B_m \Delta B) \underset{m\to\infty}{\to} 0$$

and

$$\langle \bigotimes_{i=1}^{n} X_i(A), \bigotimes_{i=1}^{n} X_i(B) \rangle_{H^{\otimes n}} = \lim_{m\to\infty} \langle \bigotimes_{i=1}^{n} X_i(A_m), \bigotimes_{i=1}^{n} X_i(B_m) \rangle_{H^{\otimes n}}.$$

Then it is enough to show that

$$\sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A_m \cap B_m^{\Pi}) \underset{m\to\infty}{\to} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi}).$$

Using the fact that for all Π and $A, B \in A^n$

$$|\mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi})| \leq \{\bigotimes_{i=1}^{n} \mu_i(A)\}^{1/2} \{\bigotimes_{i=1}^{n} \mu_i(B)\}^{1/2}$$

we have that

$$\left|\frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi}) - \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A_m \cap B_m^{\Pi})\right|$$

$$= \left|\frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi} \setminus (A_m \cap B_m^{\Pi})) - \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A_m \cap B_m^{\Pi} \setminus (A \cap B^{\Pi}))\right|$$

$$= \frac{1}{n!} \Big| \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi} \cap A_m^c) + \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A \cap B^{\Pi} \cap (B_m^{\Pi})^c) - \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A_m \cap B_m^{\Pi} \cap A^c) - \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A_m \cap B_m^{\Pi} \cap (B^{\Pi})^c) \Big|$$

$$\leq \frac{1}{n!} \sum_{\Pi} | \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n} (A \cap B^{\Pi} \cap A_m^c) |$$

$$+ \frac{1}{n!} \sum_{\Pi} | \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n} (A \cap B^{\Pi} \cap (B_m^{\Pi})^c) |$$

$$+ \frac{1}{n!} \sum_{\Pi} | \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n} (A_m \cap B_m^{\Pi} \cap A^c) |$$

$$+ \frac{1}{n!} \sum_{\Pi} | \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n} (A_m \cap B_m^{\Pi} \cap (B^{\Pi})^c) |$$

$$\leq \{ \bigotimes_{i=1}^{n} \mu_i (A \backslash A_m) \}^{\frac{1}{2}} \{ \bigotimes_{i=1}^{n} \mu_i (B) \}^{\frac{1}{2}} + \{ \bigotimes_{i=1}^{n} \mu_i (A) \}^{\frac{1}{2}} \{ \bigotimes_{i=1}^{n} \mu_i (B \backslash B_m) \}^{\frac{1}{2}}$$

$$+ \{ \bigotimes_{i=1}^{n} \mu_i (A_m \backslash A) \}^{\frac{1}{2}} \{ \bigotimes_{i=1}^{n} \mu_i (B_m) \}^{\frac{1}{2}} + \{ \bigotimes_{i=1}^{n} \mu_i (A_m) \}^{\frac{1}{2}} \{ \bigotimes_{i=1}^{n} \mu_i (B_m \backslash B) \}^{\frac{1}{2}}$$

which goes to zero as $m \to \infty$, since $\bigotimes_{i=1}^{n} \mu_i (A \Delta A_m) \underset{m\to\infty}{\to} 0$ and $\bigotimes_{i=1}^{n} \mu_i (B_m \Delta B) \underset{m\to\infty}{\to} 0$.

Q.E.D.

Corollary 2.2.2 If $A \in A^n$, then

$$(2.2.14) \qquad \| \bigodot_{i=1}^{n} X_i (A) \|^2_{H^{\odot n}} = \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n} (A \cap A^{\Pi}) .$$

We now consider special cases for which the previous results simplify.

Symmetric sets A set $A \in A^n$ is called a symmetric set if $A^{\Pi} = A$ for all permutation Π of $\{1,\ldots,n\}$, where A^{Π} is defined in (2.2.13). The σ-field of symmetric sets of A^n (Dellacherie and Meyer (1978)) is denoted by $A^{\odot n}$. Since for $A \in A^{\odot n}$ $A \cap A^{\Pi} = A$ for all Π, we obtain the following result which is a generalization to the case of several orthogonally scattered measures of corollary to Theorem 1 on Chevet (1981).

Corollary 2.2.3 The vector measure $\bigodot_{i=1}^{n} X_i$ is an $H^{\odot n}$-valued orthogonally scattered measure on $(T^n, A^{\odot n})$ with control measure $\mu^{\odot n}$ given by

$$\mu^{\otimes n}(A) = \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \otimes \cdots \otimes \mu_{n\Pi_n}(A) \quad A \in A^{\otimes n}. \tag{2.2.15}$$

and

$$\| \overset{n}{\underset{i=1}{\otimes}} X_i(A) \|^2_{H^{\otimes n}} = \mu^{\otimes n}(A).$$

<u>Antisymmetric sets</u> Let I denote the identity permutation of $(1,2,\ldots n)$. A set $A \in A^n$ is called an <u>antisymmetric set</u> if $A \cap A^{\Pi} = \phi$ for all permutation Π of $(1,\ldots,n)$ distinct from I. For this kind of set Corollary 2.2.2 simplifies as follows:

<u>Corollary 2.2.4</u> If $A \in A^n$ is an antisymmetric set, then

$$\| \overset{n}{\underset{i=1}{\otimes}} X_i(A) \|^2_{H^{\otimes}} = \frac{1}{n!} \overset{n}{\underset{i=1}{\otimes}} \mu_i(A). \tag{2.2.16}$$

The next result is an application of the last corollary to the case when T is an interval of the real line and the measures $\mu_1,\ldots,\mu_n$ are non-atomic. This will be a useful result in the next section.

<u>Corollary 2.2.5</u> Let $T \subseteq \mathbb{R}$ be an interval of the real line, $A = B(T)$ and suppose that $\mu_1,\ldots,\mu_n$ are finite non-atomic measures on (T,A). For each permutation $\Pi = (\Pi_1,\ldots,\Pi_n)$ of $(1,\ldots,n)$ let

$$T^n_{\Pi} = \{(t_1,\ldots,t_n) \in T^n \colon t_{\Pi_1} < \cdots < T_{\Pi_n}\}. \tag{2.2.17}$$

Then for each Π T^n_{Π} is an antisymmetric set of A^n and for each $A \in A^n$

$$\overset{n}{\underset{i=1}{\otimes}} \mu_i(A) = n! \sum_{\Pi} \| \overset{n}{\underset{i=1}{\otimes}} X_i(A \cap T^n_{\Pi}) \|^2_{H^{\otimes n}}.$$

The σ-field

$$A^n_{\Pi} = A \cap T^n_{\Pi} \tag{2.2.18}$$

is called the <u>σ-field of antisymmetric</u> subsets of T^n_{Π} corresponding to Π.

Proof First note that if Π and Π^* are two distinct permutations of $(1,\ldots,n)$ then $T^n_\Pi \cap T^n_{\Pi^*} = \phi$. Let $S^n = \bigcup_\Pi T^n_\Pi$ where the union is taken over all permutations Π of $(1,\ldots,n)$. Then

$$(S^n)^c = \{(t_1,\ldots,t_n) \in T^n\colon\ t_i = t_j \text{ for some } i \neq j\}$$

and since the measures $\mu_1,\ldots,\mu_n$ are non-atomic

$$\bigotimes_{i=1}^n \mu_i((S^n)^c) = 0.$$

Hence

$$\bigotimes_{i=1}^n \mu_i(A) = \sum_\Pi \bigotimes_{i=1}^n \mu_i(A \cap T^n_\Pi) \qquad A \in A^n.$$

But for each permutation Π, T^n_Π is an antisymmetric set; then using Corollary 2.2.4

$$\left\| \bigotimes_{i=1}^n X_i(A \cap T^n_\Pi) \right\|^2_{H^{\otimes n}} = \frac{1}{n!} \bigotimes_{i=1}^n \mu_i(A \cap T^n_\Pi) \qquad A \in A^n$$

and hence

$$\bigotimes_{i=1}^n \mu_i(A) = n! \sum_\Pi \left\| \bigotimes_{i=1}^n X_i(A \cap T^n_\Pi) \right\|^2_{H^{\otimes n}} \qquad A \in A^n.$$

Q.E.D.

We now take into consideration a concept that plays an important role in the theory of integration with respect to vector valued measures.

Semivariation of $\bigotimes_{i=1}^n X_i$ We first review the definition of semivariation of a bounded vector valued measure and related concepts. We follow Kussmaul (1977).

Let M be a bounded vector valued measure on a field A_0 of a set S with values in a Banach space $(E, \|\cdot\|)$. The semivariation of M is the extended nonnegative function $sv(M;\cdot)$ whose value on a set $A \in A_0$ is given by

$$sv(M;A) = \sup \left\| \sum_j \alpha_j M(A_j) \right\|$$

where the supremum is taken over all finite partitions (A_j) of A_o into disjoint sets $A_j \in A_o$, and all finite collections (α_j) of scalars α_j satisfying $|\alpha_j| \le 1$. The set function sv(M;•) is extended to the family of all subsets B of S by defining

$$sv(M;B) = \inf\{sv(M;A): B \subseteq A,\ A \in A_o\} .$$

A subset $B \subset S$ is called an M-null set if sv(M;B) = 0. The exceptional sets for M-almost everywhere convergence are these M-null sets. For two scalar valued functions f and g on S we define

$$d(f,g) = \inf_{\alpha \ge 0}\{\alpha + sv(M; |f-g| > \alpha)\} .$$

Convergence with respect to the topology generated by the pseudometric d is called convergence in M-measure.

The following result (Diestel and Uhl (1977) page 14) is known in the theory of vector valued measures as the Bartle-Dunford-Schwartz Theorem. It will play a key role in Section 2.3. We write it here adapted to the $H^{\circledast n}$-valued bounded measure $\underset{i=1}{\overset{n}{\circledast}} X_i$ of Theorem 2.2.1.

Lemma 2.2.2 There exists a finite nonnegative measure ν on A^n such that

a) $\quad \nu(A) \le sv(\underset{i=1}{\overset{n}{\circledast}} X_i;A) \qquad A \in A^n$

and

b) $\quad \lim_{\nu(A)\to 0} sv(\underset{i=1}{\overset{n}{\circledast}} X_i;A) = 0.$

We are not able to compute exact expressions for the semivariation of $\underset{i=1}{\overset{n}{\circledast}} X_i$ nor for the measure ν of Lemma 2.2.2. However, we will find very useful upper and lower bounds for $sv(\underset{i=1}{\overset{n}{\circledast}} X_i;A)$.

Lemma 2.2.3 Let $X_1,\dots,X_n$ be o.s.m.'s and $\mu_1,\dots,\mu_n$ be measures as in

Assumption 2.2.1. Then

$$\text{(2.2.19)} \qquad sv\left(\bigotimes_{i=1}^{n} X_i ; A\right) \le \{\mu_1 \otimes \cdots \otimes \mu_n(A)\}^{1/2} .$$

Proof. Let $(\alpha_j)_{j=1}^m$ be real numbers such that $|\alpha_j| \le 1$ and let $A_1, \ldots, A_m$ be disjoint elements in A^n such that $\bigcup_{j=1}^m A_j = A$ for $m \ge 1$. Then from (2.2.9) and Theorem 2.1.3

$$\left\|\sum_{j=1}^m \alpha_j \bigotimes_{i=1}^n X_i(A_j)\right\|^2_{H^{\otimes n}} = \left\|\sigma^n_{\otimes}\left(\sum_{j=1}^m \alpha_j \bigotimes_{i=1}^n X_i(A_j)\right\|^2_{H^{\otimes n}} \le$$

$$\left\|\sum_{j=1}^m \alpha_j \bigotimes_{i=1}^n X_i(A_j)\right\|^2_{H^{\otimes n}} = \sum_{j=1}^m \sum_{k=1}^m \alpha_j \alpha_k < \bigotimes_{i=1}^n X_i(A_j), \bigotimes_{i=1}^n X_i(A_k)>_{H^{\otimes n}}$$

$$= \sum_{j=1}^m \alpha_j^2 \, \mu_1 \otimes \cdots \otimes \mu_n(A_j) \le \sum_{j=1}^m \mu_1 \otimes \cdots \otimes \mu_n(A_j) = \mu_1 \otimes \cdots \otimes \mu_n(A)$$

and hence (2.2.19) follows.

Q.E.D.

To obtain a lower bound for the semivariation of $\bigotimes_{i=1}^n X_i$ we have to assume an additional condition on the measurable space (T, A) and on the control measures $\mu_1, \ldots, \mu_n$.

Lemma 2.2.4 Let $T \subseteq \mathbb{R}$ be an interval of the real line, $A = \mathcal{B}(T)$ and $X_1, \ldots, X_n$ be o.s.m.'s as in Assumption 2.2.1 with finite non-atomic control measures $\mu_1, \ldots, \mu_n$. Then for $A \in A^n$

$$\text{(2.2.20)} \qquad \frac{1}{n!}\left\{\bigotimes_{i=1}^n \mu_i(A)\right\}^{1/2} \le sv\left(\bigotimes_{i=1}^n X_i ; A\right) \le \left\{\bigotimes_{i=1}^n \mu_i(A)\right\}^{1/2} .$$

Proof By Corollary 2.2.5, if $A \in A^n$

$$\bigotimes_{i=1}^n (A) = n! \sum_{\Pi} \left\|\bigotimes_{i=1}^n X_i(A \cap T^n_{\Pi})\right\|^2_{H^{\otimes n}}$$

where for each permutation Π of $(1, \ldots, n)$ T^n_{Π} is defined in (2.2.17).

Next from Proposition 11 of Diestel and Uhl (1977), giving a lower bound for the semivariation of a vector valued measure,

$$\sup\{\|\overset{n}{\underset{i=1}{\otimes}} X_i(B)\|_{H^{\otimes n}} : A \supseteq B \in A^n\} \le sv(\overset{n}{\underset{i=1}{\otimes}} X_i;A).$$

Then
$$n! \sum_{\Pi} \|\overset{n}{\underset{i=1}{\otimes}} X_i(A\cap T^n_{\Pi})\|^2_{H^{\otimes n}} \le n! \sum_{\Pi} \{sv(\overset{n}{\underset{i=1}{\otimes}} X_i;A)\}^2$$

$$= (n!)^2\{sv(\overset{n}{\underset{i=1}{\otimes}} X_i;A)\}^2$$

and therefore

$$\frac{1}{n!}\{\overset{n}{\underset{i=1}{\otimes}} \mu_i(A)\}^{\frac{1}{2}} \le sv(\overset{n}{\underset{i=1}{\otimes}} X_i;A).$$

The upper bound in (2.2.20) follows from the last lemma.

Q.E.D.

The next two results are consequences of the above lemma. They characterize convergence in $\overset{n}{\underset{i=1}{\otimes}} X_i$-measure in terms of the control measures $\mu_1,\ldots,\mu_n$.

Corollary 2.2.6 *Under the asuumptions of Lemma 2.2.4, for a sequence* $\{A_m\}_{m\ge1}$ in A^n, $sv(\overset{n}{\underset{i=1}{\otimes}} X_i;A_m) \underset{m\to\infty}{\to} 0$ if and only if $\overset{n}{\underset{i=1}{\otimes}} \mu_i(A_m) \underset{m\to\infty}{\to} 0$.

Corollary 2.2.7 Under the assumptions of Lemma 2.2.4, a sequence of real A^n-measurable functions $(f_m)_{m\ge1}$ converges in $\overset{n}{\underset{i=1}{\otimes}} X_i$-measure to real valued function f on T^n if and only if f_m converges to f in $\overset{n}{\underset{i=1}{\otimes}} \mu_i$-measure.

Orthogonality We now study a special property of symmetric tensor product measures. We assume that for each $n \ge 1$ we have $X_1,\ldots,X_n$ orthogonally scattered measures as in Assumption 2.2.1, all taking values in the same Hilbert space H.

The Hilbert Exponential space of H (Guichardet (1972)) is the Hilbert space

$$\text{(2.2.21)} \qquad \mathrm{EXP}(H) = \sum_{n\geq 0} \oplus H^{\circledcirc n} \qquad (H^{\circledcirc 0} \equiv \mathbb{R})$$

that is, the set of all sequences $\underline{x} = (x_n)$ $x_n \in H^{\circledcirc n}$ $n\geq 0$

such that $\sum_{n=0}^{\infty} \| x_n \|^2_{H^{\circledcirc n}} < \infty$, with inner product

$$\text{(2.2.22)} \qquad \langle \underline{x}, \underline{y} \rangle_e = \sum_{n=0}^{\infty} \langle x_n, y_n \rangle_{H^{\circledcirc n}} .$$

Since for each $n \geq 0$ $H^{\circledcirc n}$ may be seen as a subspace of EXP(H) ($(0, \ldots, x_n, \ldots)$ $x_n \in H^{\circledcirc n}$), then for each $n \geq 1$

$$\overset{n}{\underset{i=1}{\circledcirc}} X_i : A^n \to \mathrm{EXP}(H)$$

is a vector valued measure with values in EXP(H). Therefore symmetric tensor product measures of different orders may be realized as taking values in the same Hilbert space, viz, EXP(H) and we have the following result.

<u>Lemma 2.2.5</u> If $n_1 \neq n_2$, then for all $A_1 \in A^{n_1}$ and $A_2 \in A^{n_2}$, $\overset{n_1}{\underset{i=1}{\circledcirc}} X_i(A_1)$ is orthogonal to $\overset{n_2}{\underset{i=1}{\circledcirc}} X_i(A_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_e$ in EXP(H). The proof follows since $\overset{n_1}{\underset{i=1}{\circledcirc}} X_i$ and $\overset{n_2}{\underset{i=1}{\circledcirc}} X_i$ are $H^{\circledcirc n_1}$ and $H^{\circledcirc n_2}$ valued respectively and $H^{\circledcirc n_1}$ is orthogonal to $H^{\circledcirc n_2}$ in EXP(H) if $n_1 \neq n_2$.

<u>The n^{th} symmetric tensor product measure of an o.s.m. with itself</u> To conclude this section we turn to the special case of symmetric tensor product measures of an orthogonally scattered measure with itself. We show how previous results are simplified and other new results can be obtained. We take X to be an o.s.m. defined on an arbitrary measurable space (T,A), with values in a real separable Hilbert space H and finite control measure μ (which is not assumed to be nonatomic). We may take H to be H_X, the linear subspace

generated by X as in (2.1.3). For $n \geq 1$ let $X^{\circledast n}$ be the n^{th} symmetric tensor product measure on $\mathcal{A}^n$ with values in $EXP(H_X)$ (or $H_X^{\circledast n}$) given by Theorem 2.2.1, and let $\mu^{\circledast n}$ denote the n-fold product measure of μ with itself. The main properties of $X^{\circledast n}$ are summarized in the next result in which some possible simplifications of earlier results are shown.

<u>Proposition 2.2.1</u> a) For $A_1,\ldots,A_n \in \mathcal{A}$ $n \geq 1$.

$$X^{\circledast n}(A_1\times\ldots\times A_n) = X_1(A_1)\circledast\ldots\circledast X_n(A_n).$$

b) If $A \in \mathcal{A}^n$ and $B \in \mathcal{A}^m$

$$<X^{\circledast n}(A), X^{\circledast m}(B)>_e = \delta_{nm} <X^{\circledast n}(A), X^{\circledast n}(B)>_{H_X^{\circledast n}}$$

$$= \delta_{nm} \frac{1}{n!} \sum_{\Pi} \mu^{\circledast n}(A\cap B^{\Pi}) = \delta_{nm} \frac{1}{n!} \sum_{\Pi} \mu^{\circledast n}(A^{\Pi}\cap B).$$

c) $$||X^{\circledast n}(A)||^2_{H_X^{\circledast n}} = \frac{1}{n!} \sum_{\Pi} \mu^{\circledast n}(A\cap A^{\Pi}) \leq \mu^{\circledast n}(A) \qquad A \in \mathcal{A}^n \quad n \geq 1.$$

d) The vector measure $X^{\circledast n}$ is an $H_X^{\circledast n}$-valued orthogonally scattered measure on $(T^n, \mathcal{A}^{\circledast n})$ with control measure $\mu^{\circledast n}$.

e) If $A \in \mathcal{A}^n$ is an antisymmetric set

$$||X^{\circledast n}(A)||^2_{H_X^{\circledast n}} = \frac{1}{n!} \mu^{\circledast n}(A).$$

The proof of (a) follows from Theorem 2.2.1. Lemmas 2.2.1 and 2.2.5 imply (b) and Corollaries 2.2.2, 2.2.3 and 2.2.4 imply (c), (d) and (e) respectively.

In particular, Lemma 2.2.4 applies to $X^{\circledast n}$. However, in that lemma we have assumed that T is an interval of the real line and the control measure is nonatomic. In the case of $X^{\circledast n}$ it is possible to improve Lemma 2.2.4 and

compute an exact expression for the semivariation of $X^{\otimes n}$, without assuming that $T \subseteq \mathbb{R}$ or μ is nonatomic.

Lemma 2.2.6 Let X be an o.s.m. on an arbitrary measurable space $(T,\mathcal{A})$ with values in H_X and finite control measure μ (not necessarily nonatomic). Then for $A \in \mathcal{A}^n$

$$(2.2.23) \qquad sv(X^{\otimes n};A) = \{\frac{1}{n!} \sum_{\Pi} \mu^{\otimes n}(A \cap A^{\Pi})\}^{\frac{1}{2}} = \| X^{\otimes n}(A) \|_{H_X^{\otimes n}} .$$

Proof From (c) in the last proposition and the definition of $sv(X^{\otimes n};A)$

$$(2.2.24) \qquad \frac{1}{n!} \sum_{\Pi} \mu^{\otimes n}(A \cap A^{\Pi}) = \| X^{\otimes n}(A) \|^2_{H_X^{\otimes n}} \le sv(X^{\otimes n};A) .$$

On the other hand if $|\alpha_i| \le 1$ $i=1,\dots,m$ are real numbers and $A_1,\dots,A_m$ are disjoint sets in $\mathcal{A}^n$, $\bigcup_{i=1}^{m} A_i = A$, then for each permutation Π of $(1,\dots,n)$ $A_1^{\Pi},\dots,A_m^{\Pi}$ are disjoint sets in $\mathcal{A}^n$ and $A^{\Pi} = \bigcup_{i=1}^{m} A_i^{\Pi}$. Thus, using (b) in the last proposition and the fact that $\mu^{\otimes n}$ is a positive measure on $\mathcal{A}^n$ we obtain

$$\| \sum_{i=1}^{m} \alpha_i X^{\otimes n}(A_i) \|^2_{H_X^{\otimes n}} = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \langle X^{\otimes n}(A_i), X^{\otimes n}(A_j) \rangle_{H_X^{\otimes n}}$$

$$= \frac{1}{n!} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \sum_{\Pi} \mu^{\otimes n}(A_i \cap A_j^{\Pi}) \le \frac{1}{n!} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{\Pi} \mu^{\otimes n}(A_i \cap A_j^{\Pi})$$

$$= \frac{1}{n!} \sum_{\Pi} \mu^{\otimes n}(A \cap A^{\Pi}) = \| X^{\otimes n}(A) \|^2_{H_X^{\otimes n}} .$$

Therefore

$$sv(X^{\otimes n};A)^2 \le \frac{1}{n!} \sum_{\Pi} \mu^{\otimes n}(A \cap A^{\Pi})$$

and the lemma follows by using (2.2.24).

Q.E.D

As one may expect Corollaries 2.2.6 and 2.2.7 can be improved for $X^{\otimes n}$.

Corollary 2.2.8 Under the hypotheses of Lemma 2.2.6 for $A \in A^n$

$$\{\frac{1}{n!} \mu^{\otimes n}(A)\}^{\frac{1}{2}} \le sv(X^{\otimes n};A) \le \{\mu^{\otimes n}(A)\}^{\frac{1}{2}} \tag{2.2.25}$$

and for a sequence $(A_m)_{m\geq 1}$ in A^n $sv(X^{\otimes n};A_m) \underset{m\to\infty}{\to} 0$ if and only if $\mu^{\otimes n}(A_m) \underset{m\to\infty}{\to} 0$.

Proof Since $\mu^{\otimes n}$ is a positive measure on A^n then for each $A \in A^n$

$$\mu^{\otimes n}(A) \le \sum_{\Pi} \mu^{\otimes n}(A \cap A^{\Pi}) \le n!\mu^{\otimes n}(A)$$

and hence (2.2.25) follows from (2.2.23). The second part of the corollary follows from (2.2.25).

Q.E.D.

Corollary 2.2.9 Under the hypothesis of Lemma 2.2.6 a sequence of real valued A^n-measurable functions $\{f_m\}_{m\geq 1}$ on T^n converges to a real valued function f on T^n in $X^{\otimes n}$-measure if and only if f_m converges to f in $\mu^{\otimes n}$-measure.

The proof follows by the above corollary and the definition of convergence in M-measure for a vector valued measure M, given before Lemma 2.2.2.

2.3 Integrals with respect to the symmetric tensor product measure

We now apply the theory of integration with respect to vector valued measures (Dunford and Schwartz (1958)) to define a multiple integral

$$\int_{T^n} \dots \int f(t_1,\dots,t_n)dX_1(t_1)\dots dX_n(t_n) = \int_{T^n} f(\underline{t})d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t})$$

where f is a real valued function and $\overset{n}{\underset{i=1}{\otimes}} X_i$ is the $H^{\otimes n}$-valued measure of Theorem 2.2.1. We assume, unless otherwise stated, that H, $X_1,\dots,X_n$, μ_{ij} $i,j=1,\dots,n$ and (T,A) are as in Assumption 2.2.1 of the beginning of Section 2.2.

We begin by presenting a definition and a proposition from the theory of integration with respect to bounded vector valued measures. They are given, for example, in the book by Kussmaul (1977) (Definition 10.3 and Proposition 10.4). We shall write them here using the notation for the vector valued measure $\overset{n}{\underset{i=1}{\otimes}} X_i$.

Definition 2.3.1 Let $f(\underline{t})$ be an A^n-measurable simple function on T^n, that is

$$(2.3.1) \qquad f(\underline{t}) = \sum_{j=1}^{k} \alpha_j 1_{A_j}(\underline{t}) \qquad \underline{t} = (t_1,\ldots,t_n)$$

where $\alpha_j \in \mathbb{R}$ $j=1,\ldots,k$ and $A_1,\ldots,A_k$ are disjoint elements in A^n. The integral of f with respect to $\overset{n}{\underset{i=1}{\otimes}} X_i$, denoted by $\int_{T^n} f(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t})$, is the element of $H^{\otimes n}$ given by

$$(2.3.2) \qquad \int_{T^n} f(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) = \sum_{j=1}^{k} \alpha_j \overset{n}{\underset{i=1}{\otimes}} X_i(A_j).$$

A real valued function f on T^n is said to be $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable if there exists a sequence $\{f_m\}_{m\geq 1}$ of A^n-measurable simple functions on T^n such that

$$(2.3.3) \qquad f_m \text{ converges to f in } \overset{n}{\underset{i=1}{\otimes}} X_i\text{-measure and}$$

$$(2.3.4) \qquad \lim_{sv(\overset{n}{\underset{i=1}{\otimes}} X_i\, ;A)\to 0} \int_{T^n} (1_A f_m)(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) = 0$$

uniformly in $m = 1,2,\ldots$, i.e.: for each $\varepsilon > 0$ there exists $\delta > 0$ (independent of m) such that for every set A for which $sv(\otimes_{i=1}^{n} X_i\,;A) < \delta$ we have

$$\left|\int_{T^n} 1_A f_m(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t})\right| < \varepsilon \quad \text{for } m=1,2,\ldots .$$

We denote by $L_1(\overset{n}{\underset{i=1}{\otimes}} X_i)$ the class of all $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable functions.

Proposition 2.3.1 Let $f(\underline{t})$ $\underline{t} \in T^n$ be a $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable function and $\{f_m\}_{m\geq 1}$ be a sequence of A^n-simple functions satisfying (2.3.3) and (2.3.4).

Then for every $A \in \mathcal{A}^n$ $(1_A f)(\underline{t})$ is $\underset{i=1}{\overset{n}{\otimes}} X_i$-integrable and the sequence

$$\{\int_{T^n} (1_A f_m)(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t})\}_{m\geq 1}$$

converges to an element in $H^{\otimes n}$ uniformly in $A \in \mathcal{A}^n$. The element

$$\int_{T^n} (1_A f)(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t}) = \lim_{m\to\infty} \int_{T^n} (1_A f_m)(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t}) \tag{2.3.5}$$

is called the integral of f with respect to $\underset{i=1}{\overset{n}{\otimes}} X_i$ over the set A.

Sometimes we will use the following notation

$$I_n(f;\ X_1,\ldots,X_n) = \int_{T^n} f(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t})$$

and

$$\int_A f(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t}) = \int_{T^n} (1_A f)(\underline{t})\, d \underset{i=1}{\overset{n}{\otimes}} X_i(\underline{t}) .$$

We now obtain a sufficient condition for the $\underset{i=1}{\overset{n}{\otimes}} X_i$-integrability of a function f in terms of the control measures $\mu_1,\ldots,\mu_n$.

<u>Theorem 2.3.1</u> If $f \in L^2(T^n, \mathcal{A}^n, \underset{i=1}{\overset{n}{\otimes}} \mu_i)$ then f is $\underset{i=1}{\overset{n}{\otimes}} X_i$-integrable.

<u>Proof</u> Since $f \in L^2(T^n, \mathcal{A}^n, \underset{i=1}{\overset{n}{\otimes}} \mu_i)$, there exists a sequence $\{f_m\}_{m\geq 1}$ of $\mathcal{A}^n$-measurable simple functions such that $|f_m| \leq |f|$ a.e. $\underset{i=1}{\overset{n}{\otimes}} \mu_i$ for $m \geq 1$ and f_m converges to f in $L^2(T^n, \mathcal{A}^n, \underset{i=1}{\overset{n}{\otimes}} \mu_i)$. Then f_m converges to f in $\underset{i=1}{\overset{n}{\otimes}} \mu_i$-measure and by Lemma 2.2.3 f_m converges to f in $\underset{i=1}{\overset{n}{\otimes}} X_i$-measure. Thus condition (2.3.3) in Definition 2.3.1 is satisfied.

Next, for $A \in \mathcal{A}^n$ $1_A(\underline{t})(f_m(\underline{t}) - f_k(\underline{t}))$ is a simple function for all $m, k \geq 1$, i.e.

$$1_A(\underline{t})(f_m(\underline{t}) - f_k(\underline{t})) = \sum_{j=1}^{\ell} \alpha_j 1_{A_j}(\underline{t})$$

for some $\alpha_j \in \mathbb{R}$ and $A_1,\ldots,A_\ell$ disjoint elements in $\mathcal{A}^n$. Then by definition

of $\overset{n}{\underset{i=1}{\otimes}} X_i$ (Theorem 2.2.1)

$$\left\| \int_{T^n} (1_A(f_m - f_k))(\underline{t}) d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) \right\|^2_{H^{\otimes n}} = \left\| \sum_{j=1}^{\ell} \alpha_j \overset{n}{\underset{i=1}{\otimes}} X_i(A_j) \right\|^2_{H^{\otimes n}}$$

$$= \left\| \sigma^n_{\otimes} \left(\sum_{j=1}^{\ell} \alpha_j \overset{n}{\underset{i=1}{\otimes}} X_i(A_j) \right) \right\|^2_{H^{\otimes n}} \leq \left\| \sum_{j=1}^{\ell} \alpha_j \overset{n}{\underset{i=1}{\otimes}} X_i(A_j) \right\|^2_{H^{\otimes n}}$$

and using Theorem 2.1.3

$$\left\| \sum_{j=1}^{\ell} \alpha_j \overset{n}{\underset{i=1}{\otimes}} X_i(A_j) \right\|^2_{H^{\otimes n}} = \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \alpha_{j_1} \alpha_{j_2} \left\langle \overset{n}{\underset{i=1}{\otimes}} X_i(A_{j_1}), \overset{n}{\underset{i=1}{\otimes}} X_i(A_{j_2}) \right\rangle_{H^{\otimes n}}$$

$$= \sum_{j=1}^{\ell} \alpha_j^2 \overset{n}{\underset{i=1}{\otimes}} \mu_i(A_j) = \int_{T^n} 1_A(\underline{t}) |f_m(\underline{t}) - f_k(\underline{t})|^2 d \overset{n}{\underset{i=1}{\otimes}} \mu_i(\underline{t})$$

which goes to zero as $m,k \to \infty$ for $A \in A^n$ because $\{f_m\}_{m \geq 1}$ is a Cauchy sequence in $L^2(T^n, A^n, \otimes_{i=1}^n \mu_i)$. Therefore for each $A \in A^n$ $\lim_{m \to \infty} \int_A f_m(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t})$ exists.

Next, since for each $A \in A^n$ and $m \geq 1$ $1_A f_m$ is a simple function, i.e. there exist α_j $j=1,\ldots,\ell$ and disjoint elements $A_1,\ldots,A_\ell$ of A^n such that

$$1_A f_m(\underline{t}) = \sum_{j=1}^{\ell} \alpha_j^m 1_{A_j}(\underline{t}),$$

then from (2.3.2) and the definition of $sv(\overset{n}{\underset{i=1}{\otimes}} X_i; A)$

$$\left\| \int_A f_m(\underline{t}) d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) \right\|_{H^{\otimes n}} = \left\| \sum_{j=1}^{\ell} \alpha_j^m \overset{n}{\underset{i=1}{\otimes}} X_i(A) \right\|_{H^{\otimes n}}$$

$$= \| f_m \|_\infty \left\| \sum_{j=1}^{\ell} \frac{\alpha_j^m}{\| f_m \|_\infty} \overset{n}{\underset{i=1}{\otimes}} X_i(A) \right\|_{H^{\otimes n}} \leq \| f_m \|_\infty sv(\overset{n}{\underset{i=1}{\otimes}} X_i; A)$$

where

$$\| f_m \|_\infty = \sup_{\underline{t} \in T^n} \| f_m(\underline{t}) \| = \max(|\alpha_1^m|, \ldots, |\alpha_\ell^m|).$$

Hence we have that for each $m \geq 1$ $\int_{(\cdot)} f_m(\underline{t}) d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t})$ is $sv(\overset{n}{\underset{i=1}{\otimes}} X_i, \cdot)$-continuous. Then by the Vitali-Hahn-Saks Theorem (Dunford and Schwartz (1958)) and Lemma 2.2.2 we have that

$$\left\| \int_{T^n} (1_A f_m)(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(t) \right\|_{H^{\otimes n}} \to 0 \quad \text{as } sv(\overset{n}{\underset{i=1}{\otimes}} X_i; A) \to 0$$

uniformly in $m=1,2,\dots$. Then condition (2.3.4) in Definition 2.3.1 is satisfied and hence f is $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable.

Q.E.D.

Under additional conditions on $(T,\mathcal{A})$ and the control measures $\mu_1,\dots,\mu_n$ we are able to give a converse of Theorem 2.3.1.

Theorem 2.3.2 Let $T \subseteq \mathbb{R}$ be an interval of the real line, $\mathcal{A} = \mathcal{B}(T)$ and suppose that $\mu_1,\dots,\mu_n$ are finite non-atomic measures on $(T,\mathcal{A})$. Then a real valued function f on T^n is $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable if and only if $f \in L^2(T^n, \mathcal{A}^n, \overset{n}{\underset{i=1}{\otimes}} \mu_i)$.

Proof Sufficiency follows from Theorem 2.3.1. So assume that f is $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable, i.e. there exists a sequence $\{f_m\}_{m\geq 1}$ of $\mathcal{A}^n$-measurable simple functions that satisfies conditions (2.3.3) and (2.3.4) of Definition 2.3.1. Next, since for each k,m $f_m - f_k$ is a simple function, $f_m(t) - f_k(t) = \sum_{j=1}^{\ell} \alpha_j 1_{A_j}(t)$ say where $\alpha_j \in \mathbb{R}$ $j=1,\dots,\ell$ and $A_1,\dots,A_\ell$ are disjoint elements in $\mathcal{A}^n$, then for each $A \in \mathcal{A}^n$ using Lemma 2.2.1 we have that

$$\text{(2.3.6)} \qquad \left\| \int_{T^n} 1_A (f_m - f_k)(\underline{t})\; d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) \right\|^2_{H^{\otimes n}} =$$

$$\left\langle \int_{T^n} 1_A (f_m - f_k)(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}),\ \int_{T^n} 1_A (f_m - f_k)(\underline{t})\, d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) \right\rangle_{H^{\otimes n}}$$

$$= \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \alpha_{j_1} \alpha_{j_2} \left\langle \overset{n}{\underset{i=1}{\otimes}} X_i(A_{j_1} \cap A),\ \overset{n}{\underset{i=1}{\otimes}} X_i(A_{j_2} \cap A) \right\rangle_{H^{\otimes n}}$$

$$= \frac{1}{n!} \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \alpha_{j_1} \alpha_{j_2} \sum_{\Pi} \mu_{1\Pi_1} \otimes \dots \otimes \mu_{n\Pi_n} (A_{j_1} \cap A \cap (A_{j_2} \cap A)^{\Pi})$$

$$= \frac{1}{n!} \sum_{\Pi} \int_{T^n} \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \alpha_{j_1} \alpha_{j_2}\, 1_{A_{j_1} \cap A}(\underline{t})\; 1_{(A_{j_2} \cap A)^{\Pi}}(\underline{t})\, d\mu_{1\Pi_1} \otimes \dots \otimes \mu_{n\Pi_n}(\underline{t})$$

$$=\frac{1}{n!}\sum_{\Pi}\int_{T^n} 1_{A\cap A^{\Pi}}(\underline{t})(f_m-f_k)(\underline{t})(f_m-f_k)_{\Pi}(\underline{t})d\mu_{1\Pi_1}\otimes\cdots\otimes\mu_{n\Pi_n}(\underline{t}).$$

Next, using the notation of Corollary 2.2.5, for each permutation Π, T^n_{Π} defined in (2.2.17) is an antisymmetric set, i.e. $T^n_{\Pi}\cap(T^n_{\Pi})^{\Pi^*}=\phi$ for each permutation Π^* distinct from the identity permutation Π. Then for each Π the above expression (2.3.6) simplifies as

$$\left\|\int_{T^n} 1_{T^n_{\Pi}}(f_m-f_k)(\underline{t})\, d\bigotimes_{i=1}^{n} X_i(\underline{t})\right\|^2_{H^{\otimes n}} =$$

$$\frac{1}{n!}\int_{T^n} 1_{T^n_{\Pi}}(\underline{t})(f_m(\underline{t})-f_k(\underline{t}))^2 d\mu_1\otimes\cdots\otimes\mu_n(\underline{t}).$$

But from Proposition 2.3.1, if $m,k\to\infty$

$$\left\|\int_{T^n}(1_A(f_m-f_k)(\underline{t})d\bigotimes_{i=1}^{n} X_i(\underline{t})\right\|^2_{H^{\otimes n}}$$

converges to zero uniformly in $A\in A^n$. Then if $S^n=\bigcup_{\Pi} T^n_{\Pi}$

$$\int_{T^n} 1_{S^n}(\underline{t})(f_m(\underline{t})-f_k(\underline{t}))^2 d\bigotimes_{i=1}^{n}\mu_i(\underline{t}) \to 0 \quad \text{as } m,k\to\infty$$

and since the measures $\mu_1,\ldots,\mu_n$ are non-atomic

$$\bigotimes_{i=1}^{n}\mu_i((S^n)^c) = 0$$

which implies that

$$\int_{T^n}|f_m(\underline{t})-f_k(\underline{t})|^2 d\bigotimes_{i=1}^{n}\mu_i(\underline{t}) \to 0 \quad \text{as } m,k\to\infty.$$

Thus $\{f_m\}_{m\geq 1}$ is a Cauchy sequence in $L^2(T^n,A^n,\bigotimes_{i=1}^{n}\mu_i)$ and since by Corollary 2.2.7 $f_m\to f$ in $\bigotimes_{i=1}^{n}\mu_i$-meaure, then f belongs to $L^2(T^n,A^n,\bigotimes_{i=1}^{n}\mu_i)$.

Q.E.D.

<u>Properties of $\int_{T^n} f(\underline{t})\, d\bigotimes_{i=1}^{n} X_i(\underline{t})$</u> Since the integral $\int_{T^n} f(\underline{t}) d\bigotimes_{i=1}^{n} X_i(\underline{t})$

has been constructed using the theory of integration w.r.t. vector valued measures, then this integral inherits the properties from that theory. For the sake of completeness we present here some of these properties. They are Propositions 2.3.2-2.3.5 whose proofs are given in Kussmaul (1977). We write them here using our notation even though they hold for every bounded vector valued measure.

On the other hand we are able to prove another kind of properties for $\int_{T^n} f(\underline{t})\, d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t})$ which use the special structure of symmetric tensor product of $\underset{i=1}{\overset{n}{\circledast}} X_i$ and Assumption 2.2.1. They do not necessarily hold for every vector valued measure. They are presented in Theorem 2.3.3, Lemmas 2.3.1-2.3.2 and Corollaries 2.3.1-2.3.3.

<u>Proposition 2.3.2</u> Let f be a $\underset{i=1}{\overset{n}{\circledast}} X_i$-integrable function on T^n, i.e. $f \in L_1(\underset{i=1}{\overset{n}{\circledast}} X_i)$. Then

a) If $g \in L_1(\underset{i=1}{\overset{n}{\circledast}} X_i)$ and $a,b \in \mathbb{R}$, for each $A \in A^n$ we have

$$\int_A (af(\underline{t})+bg(\underline{t}))d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t}) = a \int_A f(\underline{t})d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t}) + b \int_A g(\underline{t})d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t}) .$$

b) $\int_{(\cdot)} f(\underline{t})d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t})$ is a countable additive bounded measure on (T^n, A^n) with values in $H^{\circledast n}$ such that

$$sv(\int_{(\cdot)} f(\underline{t})d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t});A) \to 0 \quad \text{as } sv(\underset{i=1}{\overset{n}{\circledast}} X_i;A) \to 0.$$

<u>Proof</u> See Kussmaul (1977) Proposition 10.4.

<u>Proposition 2.3.3</u> Let S be the vector space of real A^n-measurable simple functions. Define for $f \in S$

$$(2.3.7) \qquad \|f\| = sv(\int_{(\cdot)} f(\underline{t})d \underset{i=1}{\overset{n}{\circledast}} X_i(\underline{t});T^n) .$$

Then

a) $\|f\|$ is a norm on S.

b) Let ν be the nonnegative measure on A^n given by Lemma 2.2.2. Then every $\|\cdot\|$-Cauchy sequence $\{f_m\}_{m\geq 1}$ of elements in S converges in $\bigotimes_{i=1}^{n} X_i$-measure (and hence in ν-measure) to an A^n-measurable function f on T^n.

c) The space $L_1(\bigotimes_{i=1}^{n} X_i)$ of $\bigotimes_{i=1}^{n} X_i$-integrable functions can be identified with the completion of S with respect to the norm $\|\cdot\|$ given by (2.3.7).

d) The linear operator $I_n(\cdot;\, X_1,\ldots,X_n):\ L_1(\bigotimes_{i=1}^{n} X_i) \to H^{\otimes n}$ defined by $I_n(f;\, X_1,\ldots,X_n) = \int_{T^n} f(\underline{t})\, d \bigotimes_{i=1}^{n} X_i(\underline{t})$ is continuous with norm $\|I_n\| \leq 1$.

Proof See Kussmaul (1977) Theorem 10.8.

Proposition 2.3.4 a) Let $f \in L_1(\bigotimes_{i=1}^{n} X_i)$. Then the element

$$\int_{T^n} f(\underline{t})\, d \bigotimes_{i=1}^{n} X_i(\underline{t}) \in H^{\otimes n}$$

is uniquely determined by

$$\Big\langle \int_{T^n} f(\underline{t})\, d \bigotimes_{i=1}^{n} X_i(\underline{t}), F \Big\rangle_{H^{\otimes n}} = \int_{T^n} f(\underline{t})\, d \Big\langle \bigotimes_{i=1}^{n} X_i, F\Big\rangle (\underline{t})$$

for all $F \in H^{\otimes n}$, where $\langle \bigotimes_{i=1}^{n} X_i, F\rangle(\cdot)$ is the signed meaure on A^n given by

$$\Big\langle \bigotimes_{i=1}^{n} X_i, F\Big\rangle(A) = \Big\langle \bigotimes_{i=1}^{n} X_i(A), F\Big\rangle_{H^{\otimes n}} \qquad A \in A^n.$$

b) An A^n-measurable function on T^n is $\bigotimes_{i=1}^{n} X_i$-integrable if and only if for every $F \in H^{\otimes n}$ f is $\langle \bigotimes_{i=1}^{n} X_i, F\rangle$-integrable and the family

$$\Big\{\int_{T^n} f(\underline{t})\, d\Big\langle \bigotimes_{i=1}^{n} X_i, F\Big\rangle(\underline{t}) :\ \|F\|_{H^{\otimes n}} \leq 1 \quad F \in H^{\otimes n}\Big\}$$

is weakly sequentially compact.

Proof See Kussmaul (1977) Corollaries 1 and 2, page 107.

Proposition 2.3.5 (Lebesgue Dominated Convergence Theorem).

Let $\{f_m\}_{m\geq 1}$ be a sequence of $\circledast_{i=1}^{n} X_i$ - integrable functions which converges to a function f in $\circledast_{i=1}^{n} X_i$ - measure and $|f_m| \leq g \quad m \geq 1$ where g is a $\circledast_{i=1}^{n} X_i$- integrable function. Then f is $\circledast_{i=1}^{n} X_i$ - integrable, f_m converges to f in the $L_1(\circledast_{i=1}^{n} X_i)$ - norm of Proposition 2.3.3 and

$$\int_{T^n} f(\underline{t})d \circledast_{i=1}^{n} X_i(\underline{t}) = \lim_{m\to\infty} \int_{T^n} f_m(\underline{t})d \circledast_{i=1}^{n} X_i(\underline{t}).$$

Proof See Kussmaul (1977) Corollary 3, page 108.

In all the above properties (Propositions 2.3.2-2.3.5) we have only used the fact that $\circledast_{i=1}^{n} X_i$ is a bounded vector valued measure. Now we shall use the special structure of symmetric tensor product of $\circledast_{i=1}^{n} X_i$ and the hypotheses in Assumption 2.2.1 to show additional properties of the integral $\int_{T^n} f(\underline{t})d \circledast_{i=1}^{n} X_i(\underline{t})$.

Our first result gives an expression for the inner product of two integrals and consequently for their norm. We first introduce some new notation: Let μ_o be a σ-finite non-negative measure on (T,A) such that $\mu_{ij} << \mu_o$ $i,j=1,\ldots,n$ $\left(\mu_o = \sum_{i=1}^{n} \mu_i \text{ for example}\right)$ and $R(s) = (r_{ij}(s))$ be the non-negative definite matrix a.e. $d\mu_o$ given by (2.2.2), that is

$$r_{ij}(s) = \frac{d\mu_{ij}}{d\mu_o}(s) \qquad \text{a.e. } d\mu_o \;.$$

For each $\underline{t} = (t_1,\ldots,t_n) \in \mathbb{R}^n$, let

$$R^{\otimes n}(\underline{t}) = R(t_1)\hat{\otimes}\ldots\hat{\otimes}R(t_n) \tag{2.3.8}$$

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where $\hat{\otimes}$ denotes the Kronecker product for matrices, i.e. if $A = (a_{ij})$ is an $(m\times n)$ matrix and $B = (b_{k\ell})$ is a $p\times q$ matrix, then $A\hat{\otimes}B$ is the $(mp\times nq)$ matrix

$$A\hat{\otimes}B = \begin{bmatrix} a_{11}B & a_{12}B \dots a_{1n}B \\ a_{21}B & a_{22}B \dots a_{2n}B \\ \vdots & \\ a_{m1}B & a_{m2}B \dots a_{mn}B \end{bmatrix} .$$

Then $R^{\otimes n}(\underline{t})$ defined in (2.3.8) is an $(n^n\times n^n)$ non-negative definite matrix a.e. $d\mu_o^n$.

Let $(\underline{e}_i)_{i=1}^n$ be the canonical basis in $\mathbb{R}^n$. For each permutation $\Pi = (\Pi_1,\dots,\Pi_n)$ of $(1,\dots,n)$ let

(2.3.9) $$e^{\Pi}_{\otimes n} = \underline{e}_{\Pi_1}\hat{\otimes}\dots\hat{\otimes}\underline{e}_{\Pi_n}$$

and for a given real valued function f on T^n define the $(\mathbb{R}^n)^{\otimes n}$-valued functions on T^n

(2.3.10) $$f^{\Pi}_{\otimes n}(\underline{t}) = f(\underline{t}_{\Pi})\, e^{\Pi}_{\otimes n}$$

and

(2.3.11) $$f_{\otimes n}(\underline{t}) = \frac{1}{n!} \sum_{\Pi} f^{\Pi}_{\otimes n}(\underline{t}).$$

We shall denote by $f_{\otimes n}(\underline{t})'$ the transpose of $f_{\otimes n}(\underline{t})$.

Using the above notation we now establish the next result.

<u>Theorem 2.3.3</u> a) If $f,g \in L^2(T^n, A^n, \overset{n}{\underset{i=1}{\otimes}} \mu_i)$

(2.3.12) $$\langle \int_{T^n} f(\underline{t})d \overset{n}{\underset{i=1}{\circledcirc}} X_i(\underline{t}), \int_{T^n} g(\underline{t})d \overset{n}{\underset{i=1}{\circledcirc}} X_i(\underline{t})\rangle_{H^{\otimes n}} =$$

$$\int_{T^n} f_{\otimes n}(\underline{t})'\, R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}).$$

b) If $f \in L^2(T^n, A^n, \overset{n}{\underset{i=1}{\otimes}} \mu_i)$

$$\| \int_{T^n} f(\underline{t}) d \overset{n}{\underset{i=1}{\otimes}} X_i(\underline{t}) \|^2_{H^{\otimes n}} = \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) f_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t})$$

$$\leq \int_{T^n} |f(\underline{t})|^2 \, d \overset{n}{\underset{i=1}{\otimes}} \mu_i(\underline{t}).$$

<u>Proof</u> First we will show that for $f \in L^2(T^n, A^n, \overset{n}{\underset{i=1}{\otimes}} \mu_i)$

$$\int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) f_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}) < \infty. \tag{2.3.13}$$

Since $\int_{T^n} \gamma(\underline{t})' R^{\otimes n}(\underline{t}) \beta(\underline{t}) d\mu_o^n(\underline{t})$ is a semi-inner product in the space of A^n-measurable $(\mathbb{R}^n)^{\otimes n}$-valued functions on T^n, it follows by the triangle inequality that

$$\begin{aligned} 0 &\leq \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) f_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}) \\ &= \int_{T^n} \Big(\frac{1}{n!} \sum_{\Pi} f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n}\Big)' R^{\otimes n}(\underline{t}) \Big(\frac{1}{n!} \sum_{\Pi} f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n}\Big) d\mu_o^n(\underline{t}) \\ &\leq \Big\{\frac{1}{n!} \sum_{\Pi} \Big\{\int_{T^n} (f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n})' R^{\otimes n}(\underline{t}) (f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n}) d\mu_o^n(\underline{t})\Big\}^{\frac{1}{2}}\Big\}^2 . \end{aligned} \tag{2.3.14}$$

Next using (2.3.8), (2.3.10), the transformation theorem and the fact that $r_{ii}(s) = \frac{d\mu_i}{d\mu_o}(s)$ a.e. $d\mu_o$, we have that for each permutation $\Pi = (\Pi_1, \ldots, \Pi_n)$

$$\begin{aligned} &\int_{T^n} (f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n})' R^{\otimes n}(\underline{t}) (f(\underline{t}_{\Pi}) e^{\Pi}_{\otimes n}) d\mu_o^n(\underline{t}) \\ &= \int_{T^n} f^2(\underline{t}_{\Pi}) \; r_{\Pi_1\Pi_1}(t_1) \ldots r_{\Pi_n\Pi_n}(t_n) \; d\mu_o^n(\underline{t}) \\ &= \int_{T^n} f^2(\underline{s}) r_{11}(s_1) \ldots r_{nn}(s_n) d\mu_o^n(\underline{s}) = \int_{T^n} f^2(\underline{s}) \; \frac{d \overset{n}{\underset{i=1}{\otimes}} \mu_i}{d\mu_o^n}(\underline{s}) d\mu_o^n(\underline{s}) \\ &= \int_{T^n} |f(\underline{s})|^2 d \overset{n}{\underset{i=1}{\otimes}} \mu_i(\underline{s}). \end{aligned}$$

Then from (2.3.14) we obtain that if $f \in L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$

$$\text{(2.3.15)} \qquad \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) f_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}) \le$$

$$\int_{T^n} |f(\underline{t})|^2 d \bigotimes_{i=1}^{n} \mu_i(\underline{t}) < \infty$$

which proves (2.3.13) and the inequality in (b).

Thus applying Cauchy-Schwarz inequality we have that if $f, g \in L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$, then

$$\int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}) < \infty.$$

Now we shall show that (2.3.12) holds if f and g are A^n-measurable real simple functions, that is

$$f(\underline{t}) = \sum_{k=1}^{m} a_k 1_{A_k}(\underline{t})$$

$$g(\underline{t}) = \sum_{k=1}^{m} b_k 1_{A_k}(\underline{t})$$

where $a_{k_1} b_k \in \mathbb{R}$ $k=1,\dots,m$ and $A_1,\dots,A_k$ are disjoint sets in A^n. Then using the definition of the integral $\int_{T^n} f(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t})$ for simple functions and Lemma 2.2.1

$$< \int_{T^n} f(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t}), \int_{T^n} g(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t}) >_{H^{\otimes n}}$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{m} a_j b_k < \bigotimes_{i=1}^{n} X_i(A_j), \bigotimes_{i=1}^{n} X_i(A_k) >_{H^{\otimes n}}$$

$$= \frac{1}{n!} \sum_{j=1}^{m} \sum_{k=1}^{m} a_j b_k \sum_{\Pi} \mu_{1\Pi_1} \otimes \dots \otimes \mu_{n\Pi_n} (A_j \cap A_k^{\Pi})$$

$$= \frac{1}{n!} \sum_{\Pi} \int_{T^n} f(\underline{t}) g_{\Pi}(\underline{t}) d\mu_{1\Pi_1} \otimes \dots \otimes d\mu_{n\Pi_n}(\underline{t})$$

$$= \frac{1}{n!} \sum_{\Pi} \int_{T^n} f(\underline{t}) g_{\Pi}(\underline{t})\, r_{1\Pi_1}(t_1) \dots r_{n\Pi_n}(t_n) d\mu_o^n(\underline{t})$$

$$= \frac{1}{n!} \sum_{\Pi} \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g^{\Pi}_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t})$$

$$= \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t})$$

where $f_{\otimes n}(\underline{t}) = f^{\Pi}_{\otimes n}(\underline{t})$ for Π the identity permutation.

Next for each permutation Π, applying the transformation theorem we obtain

$$\int_{T^n} f^{\Pi}_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t})$$

$$= \frac{1}{n!} \sum_{\Pi^*} \int_{T^n} f_{\Pi}(\underline{t}) g_{\Pi^*}(\underline{t}) r_{\Pi_1 \Pi^*_1}(t_1) \cdots r_{\Pi_n \Pi^*_n}(t_n) d\mu^n_o(\underline{t})$$

$$= \frac{1}{n!} \sum_{\Pi^*} \int_{T^n} f(\underline{s}) g_{\Pi^*}(\underline{s}) r_{1\Pi^*_1}(s_1) \cdots r_{n\Pi^*_n}(s_n) d\mu^n_o(\underline{s})$$

$$= \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t}).$$

Then (2.3.12) holds for A^n-simple functions using (2.3.11).

Now suppose that $f, g \in L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$. Then there exist sequences $\{f_m\}_{m\geq 1}$, $\{g_m\}_{m\geq 1}$ of A^n-measurable simple functions such that $f_m \underset{m\to\infty}{\to} f$ and $g_m \underset{m\to\infty}{\to} g$ in $L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$ and by Theorem 2.3.1

$$\int_{T^n} f_m(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t}) \underset{m\to\infty}{\to} \int_{T^n} f(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t})$$

and

$$\int_{T^n} g_m(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t}) \underset{m\to\infty}{\to} \int_{T^n} g(\underline{t}) d \bigotimes_{i=1}^{n} X_i(\underline{t}).$$

Then it is enough to show that

$$\int_{T^n} ((f_m)_{\otimes n}(\underline{t}))' R^{\otimes n}(\underline{t}) (g_m)_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t})$$

$$\underset{m\to\infty}{\to} \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu^n_o(\underline{t}).$$

Denote $L^2(\bigotimes_{i=1}^{n} \mu_i) = L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$. Then using (2.3.15)

$$
\begin{aligned}
&\Big|\int_{T^n}((f_m)_{\otimes n}(\underline{t}))' R^{\otimes n}(\underline{t})(g_m)_{\otimes n}(\underline{t})\, d\mu_o^n(\underline{t}) - \\
&\qquad \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t})\, d\mu_o^n(\underline{t})\Big| \\
&\le \Big|\int_{T^n}((f_m)_{\otimes n}(\underline{t}) - f_{\otimes n}(\underline{t}))' R^{\otimes n}(\underline{t})((g_m)_{\otimes n}(\underline{t}) - g_{\otimes n}(\underline{t})\, d\mu_o^n(\underline{t})\Big| \\
&\quad + \Big|\int_{T^n}((f_m)_{\otimes n}(\underline{t}) - f(\underline{t}))' R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t})\, d\mu_o^n(\underline{t})\Big| \\
&\quad + \Big|\int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t})((g_m)_{\otimes n}(\underline{t}) - g_{\otimes n}(\underline{t}))\, d\mu_o^n(\underline{t})\Big| \\
&\le \|f_m - f\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} \|g_m - g\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} + \|f_m - f\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} \|g\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} \\
&\quad + \|f\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} \|g - g_m\|_{L^2(\bigotimes_{i=1}^{n}\mu_i)} \to 0 \quad \text{as } m \to \infty
\end{aligned}
$$

because $f_m \underset{m\to\infty}{\to} f$ and $g_m \underset{m\to\infty}{\to} g$ in $L^2(\bigotimes_{i=1}^{n} \mu_i)$.

(b) follows from (a) taking $f = g$.

Q.E.D.

Corollary 2.3.1 If A is an antisymmetric set in A^n then for all $f \in L^2(T^n, A^n, \bigotimes_{i=1}^{n} \mu_i)$

$$
\Big\| \int_A f(\underline{t})\, d \bigotimes_{i=1}^{n} X_i(\underline{t}) \Big\|^2_{H^{\otimes n}} = \frac{1}{n!} \int_A |f(\underline{t})|^2 d \bigotimes_{i=1}^{n} \mu_i(\underline{t}).
$$

Proof Since $A \cap A^{\Pi} = \phi$ for all Π distinct from the identity permutation Π, then using (b) in the last theorem

$$
\Big\| \int_A f(\underline{t})\, d \bigotimes_{i=1}^{n} X_i(\underline{t}) \Big\|^2_{H^{\otimes n}} = \int_A f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t}) f_{\otimes n}(\underline{t})\, d\mu_o^n(\underline{t})
$$

$$=\frac{1}{n!}\sum_{\Pi}\int_{T^n} f(\underline{t})1_A(\underline{t})f_{\Pi}(\underline{t})1_{A^{\Pi}}(\underline{t})d\mu_{1\Pi_1}\otimes\cdots\otimes d\mu_{n\Pi_n}(\underline{t})$$

$$=\frac{1}{n!}\int_A |f(\underline{t})|^2\, d\mu_1\otimes\cdots\otimes d\mu_n(\underline{t}).$$

Q.E.D.

Corollary 2.3.2 If $\mu_{ij}=\mu$ $i,j,=1,\dots,n$ and $f\in L^2(T^n,A^n,\mu^{\otimes n})$, then

$$\left\|\int_{T^n} f(\underline{t})d\bigotimes_{i=1}^{n} X_i(\underline{t})\right\|^2_{H^{\otimes n}}=\int_{T^n}|\tilde{f}(\underline{t})|^2 d\mu^{\otimes n}(\underline{t})$$

where

$$\tilde{f}(\underline{t})=\frac{1}{n!}\sum_{\Pi} f_{\Pi}(\underline{t}) \quad .$$

Proof Taking $\mu_o=\mu$, then $r_{ij}(t)=1$ all $i,j,=1,\dots,n$ and from Theorem 2.3.3 (b)

$$\left\|\int_{T^n} f(\underline{t})d\bigotimes_{i=1}^{n} X_i(\underline{t})\right\|^2_{H^{\otimes n}}=\int_{T^n} f_{\otimes n}(\underline{t})'R^{\otimes n}(\underline{t})f_{\otimes n}(\underline{t})d\mu_o^n(\underline{t})$$

$$=\frac{1}{n!}\sum_{\Pi}\int_{T^n} f(\underline{t})f_{\Pi}(\underline{t})d\mu^n(\underline{t})=\int_{T^n}|\tilde{f}(\underline{t})|^2 d\mu^{\otimes n}(\underline{t}).$$

Q.E.D.

Corollary 2.3.3 If $\mu_{ij}=\mu$ $i,j=1,\dots,n$ and f is a symmetric function in $L^2(T^n,A^n,\mu^{\otimes n})$ then

$$\left\|\int_{T^n} f(\underline{t})d\bigotimes_{i=1}^{n} X_i(\underline{t})\right\|^2_{H^{\otimes n}}=\int_{T^n}|f(\underline{t})|^2 d\mu^{\otimes n}(\underline{t}) \quad .$$

The proof follows from the last corollary since $\tilde{f}=f$.

The next lemma may be seen as a Fubini's type theorem for multiple stochastic integrals. We write

$$I_n(f;\, X_1,\dots,X_n)=\int_{T^n} f(\underline{t})d\bigotimes_{i=1}^{n} X_i(\underline{t}).$$

Lemma 2.3.1 Let $f(t_1,\dots,t_n)=f_1(t_1)\dots f_n(t_n)$ where $f_i\in L^2(T,A,\mu_i)$ $i=1,\dots,n$. Then f is $\bigotimes_{i=1}^{n} X_i$-integrable and

$$I_n(f;\, X_1,\dots,X_n)=I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n)$$

where for each $i=1,\ldots,n$, I_{X_i} is the isometry between $L^2(T,A,\mu_i)$ and H_{X_i} given by Theorem 2.1.1.

Proof By Fubini's Theorem $f(\underline{t}) = f_1(t_1)\ldots f_n(t_n)$ belongs to $L^2(T^n,A^n,\bigotimes_{i=1}^n \mu_i)$ and hence by Theorem 2.3.1 f is $\bigotimes_{i=1}^n X_i$-integrable.

First assume that each f_i is a simple function on (T,A), i.e.

$$f_i(s) = \sum_{j=1}^{k_i} a_{ij} 1_{A_{ij}}(s)$$

where $a_{ij} \in \mathbb{R}$ $j=1,\ldots,k_i$ and $A_{i1},\ldots,A_{ik_i}$ are disjoint sets in A $i=1,\ldots,n$. Then

$$f(\underline{t}) = \sum_{j_1=1}^{k_1}\ldots\sum_{j_n=1}^{k_n} a_{1j_1}\ldots a_{nj_n} 1_{A_{1j_1}\times\ldots\times A_{nj_n}}(\underline{t})$$

and by Definition 2.3.1 and Theorem 2.2.1

$$\begin{aligned} I_n(f;\, X_1,\ldots,X_n) &= \sum_{j_1=1}^{k_1}\ldots\sum_{j_n=1}^{k_n} a_{1j_1}\ldots a_{nj_n} \bigotimes_{i=1}^{n} X_i(A_{1j_1}\times\ldots\times A_{nj_n}) \\ &= \sum_{j_1=1}^{k_1}\ldots\sum_{j_n=1}^{k_n} a_{1j_1}\ldots a_{nj_n} X_i(A_{1j_1})\otimes\ldots\otimes X_n(A_{nj_n}) \\ &= \Big(\sum_{j_1=1}^{k_1} a_{1j_1} X_1(A_{1j_1})\Big)\otimes\ldots\otimes\Big(\sum_{j_n=1}^{k_n} a_{nj_n} X_n(A_{nj_n})\Big) \\ &= I_{X_1}(f_1)\otimes\ldots\otimes I_{X_n}(f_n). \end{aligned}$$

Next since each $f_i \in L^2(T,A,\mu_i)$, there exist sequences $(f_i^m)_{m\geq 1}$ $i=1,\ldots,n$ of simple functions on (T,A) such that $f_i^m \xrightarrow[m\to\infty]{} f$ in $L^2(T,A,\mu_i)$ for each $i=1,\ldots,n$. Define

$$f^m(\underline{t}) = f_1^m(t_1)\ldots f_n^m(t_n)$$

then $f^m \to f$ in $L^2(T^n,A^n,\bigotimes_{i=1}^n \mu_i)$. Next since $I_n(\cdot;X_1,\ldots,X_n)$ is a bounded

linear operator, to prove the lemma it is enough to show that

$$I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m) \underset{m\to\infty}{\to} I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n).$$

Let $\sigma_\otimes^n$ be the projection operator defined in (2.2.3), then

$$\| I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m) - I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n)\|^2_{H^{\otimes n}}$$

$$= \| \sigma_\otimes^n(I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m) - I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n))\|^2_{H^{\otimes n}}$$

$$\leq \| I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m) - I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n)\|^2_{H^{\otimes n}}$$

$$= \| I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m)\|^2_{H^{\otimes n}} + \| I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n)\|^2_{H^{\otimes n}}$$

$$- 2 < I_{X_1}(f_1^m)\otimes\cdots\otimes I_{X_n}(f_n^m),\ I_{X_1}(f_1)\otimes\cdots\otimes I_{X_n}(f_n)>_{H^{\otimes n}}$$

$$= \| I_{X_1}(f_1^m)\|^2_H\cdots\| I_{X_n}(f_n^m)\|^2_H + \| I_{X_1}(f_1)\|^2_H\cdots\| I_{X_n}(f_n)\|^2_H$$

$$- 2 < I_{X_1}(f_1^m), I_{X_1}(f_1)>_H\cdots<I_{X_n}(f_n^m),\ I_{X_n}(f_n)>_H$$

which goes to zero as $m\to\infty$ since by Theorem 2.1.1

$$\| I_{X_i}(f_i^m)\|^2_H = \| f_i^m\|^2_{L^2(\mu_i)} \underset{m\to\infty}{\to} \| f_i\|^2_{L^2(\mu_i)} = \| I_{X_i}(f_i)\|^2_H$$

for each $i=1,\ldots,n$, where $L^2(\mu_i) = L^2(T,A,\mu_i)$.

Q.E.D.

The next result gives the orthogonality of multiple integrals of different order. We use the notation of Lemma 2.2.5.

<u>Lemma 2.3.2</u> (Orthogonalty) If $n_1 \neq n_2$ and $f \in L_1(\overset{n_1}{\underset{i=1}{\otimes}} X_i)$, $g \in L_1(\overset{n_2}{\underset{i=1}{\otimes}} X_i)$ then $I_{n_1}(f; X_1,\ldots,X_{n_1})$ is orthogonal to $I_{n_2}(g; X_1,\ldots,X_{n_2})$ with respect to the inner product $<\cdot,\cdot>_e$ in EXP(H).

Proof From Definition 2.3.1 $I_{n_1}(f; X_1,\dots,X_{n_1})$ takes values in $H^{\otimes n_1}$ and $I_{n_2}(f; X_1,\dots,X_n)$ in $H^{\otimes n_2}$. Then the lemma follows since for $n_1 \neq n_2$ $H^{\otimes n_1}$ and $H^{\otimes n_2}$ are orthogonal in EXP(H).

Q.E.D.

The case of only one o.s.m. X To conclude this section we consider the special case of only one measure X, i.e. X is an orthogonally scattered measure on a measurable space (T,A) with values in H_X and control measure μ. We use the hypotheses and notation of Proposition 2.2.1 and Lemma 2.2.6. For f an $X^{\otimes n}$-integrable function we will write

$$(2.3.16) \qquad I_{n\otimes}(f) = \int_{T^n} f(\underline{t})\,dX^{\otimes n}(\underline{t}) = I_n(f;X) \ .$$

Additional properties to those given above are now presented for the integral $I_{n\otimes}(f)$. They are similar to those for the multiple Wiener integral with respect to a Gaussian random measure presented in Itô (1951).

For a real valued function f on T^n we denote by $\tilde{f}$ the symmetrization of f defined as

$$\tilde{f}(\underline{t}) = \frac{1}{n!} \sum_{\Pi} f(\underline{t}_{\Pi}) \ .$$

The exponential space $EXP(H_X)$ is defined in (2.2.21) with inner product $\langle\cdot,\cdot\rangle_e$ given by (2.2.22) with corresponding norm $\|\cdot\|_e$.

Proposition 2.3.6 Let $f,g \in L^2(\mu^{\otimes n}) = L^2(T^n,A^n,\mu^{\otimes n})$. Then

a) $I_{n\otimes}(\tilde{f}) = I_{n\otimes}(f)$.

b) $\langle I_{n\otimes}(f),\ I_{n\otimes}(g)\rangle_{H_X^{\otimes n}} = \langle f,\tilde{g}\rangle_{L^2(\mu^{\otimes n})} = \langle\tilde{f},\tilde{g}\rangle_{L^2(\mu^{\otimes n})}$.

c) $\langle I_{n\otimes}(f),\ I_{m\otimes}(g)\rangle_e = \delta_{nm}\langle\tilde{f},\tilde{g}\rangle_{L^2(\mu^{\otimes n})}$.

d) $\|I_{n\otimes}(f)\|_e^2 = \|I_{n\otimes}(f)\|^2_{H_X^{\otimes n}} = \|\tilde{f}\|^2_{L^2(\mu^{\otimes n})} \le \|f\|^2_{L^2(\mu^{\otimes n})}$.

Proof a) Let f be an elementary function of the form

(2.3.17) $$f(\underline{t}) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}(\underline{t})$$

where $a_{i_1\ldots i_n} \in \mathbb{R}$ and $A_1\ldots,A_p$ are disjoint sets in A. Then

$$\tilde{f}(\underline{t}) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}(\underline{t}_{\Pi})$$

and from Definition 2.3.1

$$I_{n\circledast}(f) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} X(A_{i_1})\circledast\ldots\circledast X(A_{i_n})$$

and

$$I_{n\circledast}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} X(A_{i_{\Pi^{-1}_{(1)}}})\circledast\ldots\circledast X(A_{i_{\Pi^{-1}_{(n)}}}) .$$

But for all $\Pi = (\Pi_{(1)},\ldots,\Pi_{(n)})$ permutation of $(1,\ldots,n)$

$$X(A_{i_1})\circledast\ldots\circledast X(A_{i_n}) = X(A_{i_{\Pi^{-1}_{(1)}}})\circledast\ldots\circledast X(A_{i_{\Pi^{-1}_{(n)}}}) .$$

Then $I_{n\circledast}(f) = I_n(\tilde{f})$ if f is an elementary function as in (2.3.17).

If $f \in L^2(T^n,A^n,\mu^{\otimes n})$ a limit argument applies, since elementary functions form a dense linear manifold in $L^2(T^n,A^n,\mu^{\otimes n})$.

b) It follows from Theorem 2.3.3 as in Corollary 2.3.2 that

$$\langle I_{n\circledast}(f),\, I_{n\circledast}(g)\rangle_{H_X^{\otimes n}} = \frac{1}{n!} \sum_{\Pi} \int_{T^n} f(\underline{t})\, g_{\Pi}(\underline{t})\, d\mu^{\otimes n}(\underline{t})$$

$$= \int_{T^n} f(\underline{t})\tilde{g}(\underline{t})\, d\mu^{\otimes n}(\underline{t}) = \langle f,\tilde{g}\rangle_{L^2(\mu^{\otimes n})}$$

and the second equality of (b) follows from (a). The proof of (c) and (d) follows from (b) and Lemma 2.3.2

Q.E.D.

We denote by $\hat{L}^2(\mu^{\otimes n})$ the subspace of $L^2(T^n, A^n, \mu^{\otimes n})$ consisting of all *symmetric functions* ($f(\underline{t}_\Pi) = f(\underline{t})$ for all Π).

<u>Proposition 2.3.7</u> (Orthogonal expansions). Let $\psi \in EXP(H_X)$. Then

$$\psi = \sum_{n=0}^{\infty} I_{n\circledast}(\tilde{f}_n) \qquad (\|\cdot\|_e\text{-convergence})$$

where $\tilde{f}_n \in \hat{L}^2(\mu^{\otimes n})$ $n \geq 1$. Moreover

$$\|\psi\|_e^2 = \sum_{n=0}^{\infty} \|f_n\|^2_{\hat{L}^2(\mu^{\otimes n})} = \sum_{n=0}^{\infty} \|I_{n\circledast}(\tilde{f}_n)\|^2_{H_X^{\circledast n}} < \infty.$$

i.e. the system of multiple integrals $\{I_{n\circledast}(\tilde{f}_n) : \tilde{f}_n \in \hat{L}^2(\mu^{\otimes n})\}$ is complete in $EXP(H_X)$.

<u>Proof</u> For $h \in H_X$ let

$$\exp \circledast(h) = (1, h, \frac{1}{\sqrt{2!}} h^{\circledast 2}, \frac{1}{\sqrt{3!}} h^{\circledast 3}, \ldots).$$

It is known (Guichardet (1972)) that $\{\exp \circledast(h) : h \in H_X\}$ generates the space $EXP(H_X)$ and therefore if $\psi \in EXP(H_X)$

$$\text{(2.3.18)} \qquad \psi = \sum_{n=0}^{\infty} \psi_n \qquad \psi_n \in H_X^{\circledast n} \quad n \geq 1 \qquad \psi_o = \text{constant}.$$

Next, let $h_i \in H_X$ $i = 1, \ldots, n$. Then by Theorem 2.1.1 $h_i = I_X(g_i)$ $g_i \in L^2(T, A, \mu)$ $i = 1, \ldots, n$ and by Lemma 2.3.1 and Proposition 2.3.6 (b)

$$h_1 \circledast \ldots \circledast h_n = I_X(g_1) \circledast \ldots \circledast I_X(g_n) = I_{n\circledast}(g_1 \ldots g_n) = I_{n\circledast}(\tilde{f}_n)$$

where $f_n = g_1 \ldots g_n$. But from Proposition 2.3.6 (b) there is an isometry between $\hat{L}^2(\mu^{\otimes n})$ and a closed subspace R_n of $H_X^{\circledast n}$ where $R_n = \{I_{n\circledast}(\tilde{f}) : f \in \hat{L}^2(\mu^{\otimes n})\}$. Then since elements of the form $h_1 \circledast \ldots \circledast h_n$ generate $H_X^{\circledast n}$, it follows by continuity of $I_{n\circledast}$ that if $\psi_n \in H_X^{\circledast n}$, then $\psi_n = I_{n\circledast}(f_n)$ where $f_n \in \hat{L}^2(\mu^{\otimes n})$. Then the proposition follows from (2.3.18).

Q.E.D.

CHAPTER III

PRODUCT STOCHASTIC MEASURES AND MULTIPLE STOCHASTIC INTEGRALS OF L^2-INDEPENDENTLY SCATTERED MEASURES

Let $(T,\mathcal{A})$ be a measurable space and $(\Omega,\mathcal{F},P)$ be a complete probability space. The real valued set function X on $(T,\mathcal{A})$ is said to be an independently scattered measure (i.s.m.) on $(T,\mathcal{A})$ if for each sequence of pairwise disjoint sets $\{A_k\}_{k\geq 1}$ in $\mathcal{A}$, $\{X(A_k)\}_{k\geq 1}$ is a sequence of independent random variables on $(\Omega,\mathcal{F},P)$ and

$$X\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} X(A_k) \qquad \text{a.s. .}$$

We say that X is an $L^2(\Omega)$-valued i.s.m if $X(A)$ belongs to $L^2(\Omega,\mathcal{F},P)$ for each $A \in \mathcal{A}$ and the above series converges in $L^2(\Omega)$. Zero mean $(E(X(A)) = 0$ $\forall A \in \mathcal{A})$ $L^2(\Omega)$-valued independently scattered measures are special cases of orthogonally scattered measures.

In this chapter we apply the results obtained in the last one to study L^2-valued product stochastic measures and multiple stochastic integrals of non-identically distributed L^2-independently scattered measures. We include the Gaussian (Section 3.1), Poisson (Section 3.2) and general L^2-independent increments process (Section 3.3) cases. Although the last situation includes the first two, we gain generality in the measurable space $(T,\mathcal{A})$ by studying them separately, apart from the fact that results presented in the first two sections are later used in Section 3.3. In each case we present the identification of the exponential space and the symmetric tensor products of the common Hilbert space where the indepen-

dently scattered measures take values. We conclude the chapter with Section 3.4 where we make comparisons with recent works in the literature including the one by Engel (1982) on the L^2-theory of products of different stochastic measures.

3.1 Gaussian random measures

In this section we consider non-identically distributed Gaussian random measures and their symmetric tensor products. We use the well-known identification of the exponential space of a Gaussian process (Neveu (1968), Kallianpur (1970)) and our results of Section 2.2 to obtain an L^2-valued product stochastic measure. Further, applying the theory of Section 2.3, we construct multiple integrals, obtaining as special cases the multiple Wiener integral of Itô (1951) and the multiple stochastic integral with dependent integrators of Fox and Taqqu (1984).

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $T \subseteq \mathbb{R}^d$ $(d \geq 1)$ a measurable subset, $\mathcal{A} = \mathcal{B}(T)$ its Borel subsets and $\mathcal{A}_c$ its relatively compact subsets. Let $W(A) = (W_1(A),\ldots,W_n(A))$ $A \in \mathcal{A}_c$ be a zero mean n-dimensional Gaussian random field on $(\Omega,\mathcal{F},P)$, such that $W(A)$ and $W(B)$ are independent if $A \cap B = \phi$, $A,B \in \mathcal{A}_c$. Then for each $i=1,\ldots,n$ W_i is a Gaussian random measure and therefore an $L^2(\Omega,\mathcal{F}^W,P)$-valued orthogonally scattered measure, where $\mathcal{F}^W = \sigma(W(A);\ A \in \mathcal{A}_c)$. Further, W_i and W_j are independent (and hence orthogonal) over disjoint sets for $i,j=1,\ldots,n$. Define

$$\mu_i(A) = E[W_i(A)]^2 \qquad A \in \mathcal{A}_c \qquad i=1,\ldots,n \tag{3.1.1}$$

$$\mu_{ij}(A\cap B) = E(W_i(A)W_j(B)) \qquad A,\ B \in \mathcal{A}_c \quad i,j=1,\ldots,n \tag{3.1.2}$$

and let

$$H = \overline{\mathrm{sp}}\ \{\underline{a}'W(A):\ \underline{a} \in \mathbb{R}^n,\ A \in \mathcal{A}_c\} \tag{3.1.3}$$

be the Gaussian space, closed subspace of $L^2(\Omega,F^W,P)$ generated by W. Then each W_i is an H-valued orthogonally scattered measure on (T,A) with control measure μ_i, which we assume finite for i=1,...,n. (If W is a Wiener process this finiteness assumption means T has to be a bounded set, since μ_i is Lebesgue measure.)

It is known (Proposition 7.3 of Neveu (1968)) that

$$\text{(3.1.4)} \qquad \mathrm{EXP}(H) \overset{\psi}{\cong} L^2(\Omega,F^W,P)$$

where for all $h \in H$

$$\text{(3.1.5)} \qquad \psi(\exp \circledast(h)) = \exp(h - \tfrac{1}{2} E(h^2))$$

$$\exp \circledast(h) = \sum_{n\geq 0} \left(\frac{1}{n!}\right)^{\frac{1}{2}} h^{\circledast n} \in \mathrm{EXP}(H)$$

and that $\{\psi(\exp \circledast(h)) : h \in H\}$ generates $L^2(\Omega,F^W,P)$.

In our first result of this chapter we obtain an L^2-valued product stochastic measure of the Gaussian random measures $W_1,\ldots,W_n$.

<u>Proposition 3.1.1</u> Let W_i i=1,...,n be Gaussian random measures on (T,A) satisfying the above conditions. Then there exists a unique $L^2(\Omega,F^W,P)$-valued measure $\overset{n}{\underset{i=1}{\circledast}} W_i$ on (T^n,A^n) such that for $A_i \in A$ i=1,...,n

$$\text{(3.1.6)} \qquad \overset{n}{\underset{i=1}{\circledast}} W_i(A_1\times\ldots\times A_n) = W_1(A_1)\circledast\ldots\circledast W_n(A_n)$$

and for $A \in A^n$

$$E\left(\overset{n}{\underset{i=1}{\circledast}} W_i(A)\right) = 0$$

$$\mathrm{VAR}\left(\overset{n}{\underset{i=1}{\circledast}} W_i(A)\right) = \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}\circledast\ldots\circledast\mu_{n\Pi_n}(A\cap A^{\Pi}).$$

<u>Proof</u> Existence and uniqueness of the $H^{\circledast n}$-valued measure $\overset{n}{\underset{i=1}{\circledast}} W_i$ follow

from Theorem 2.2.1. By (3.1.4) $\overset{n}{\underset{i=1}{\circledcirc}} W_i$ is seen as an $L^2(\Omega, F^W, P)$-valued measure. Since for each $n \geq 1$, $\overset{n}{\underset{i=1}{\circledcirc}} W_i$ is $H^{\circledcirc n}$-valued and $H^{\circledcirc n}$ is orthogonal to $H^{\circledcirc 0} \equiv \mathbb{R}$, then $E(\overset{n}{\underset{i=1}{\circledcirc}} W_i(A)) = 0$ for $A \in A^n$. The expression for the variance follows from (2.2.14) in Corollary 2.2.2.

Q.E.D.

The $H^{\circledcirc n}$-valued measure $\overset{n}{\underset{i=1}{\circledcirc}} W_i$ is an example of the symmetric tensor product measure constructed in Section 2.2. Then all results of that section may be applied to this $L^2(\Omega)$-valued product stochastic measure $\overset{n}{\underset{i=1}{\circledcirc}} W_i$.

In order to compute the symmetric tensor product measure $\overset{n}{\underset{i=1}{\circledcirc}} W_i$ for some sets $A \in A^n$ we shall use the identification of the Exponential space of a general Gaussian space studied by Neveu (1968) and Kallianpur (1970). The next well known result is Proposition 7.5 in the work of the first named author. We present it here for the sake of completeness and later reference.

<u>Proposition 3.1.2</u> (Neveu (1968)). If H is a Gaussian space and $h_1, \ldots h_k$ are orthogonal elements in H, then

$$h_1^{\circledcirc n_1} \circledcirc \ldots \circledcirc h_k^{\circledcirc n_k} = (n!)^{-\frac{1}{2}} \prod_{j=1}^{k} h_{n_j}(h_j; \sigma_j^2) \tag{3.1.7}$$

where $\sigma_j^2 = E(h_j^2)$, $n = \sum_{j=1}^{k} n_j$ and $h_m(x;\sigma^2)$ are Hermite polynomials defined by

$$h_m(x;\sigma^2) = (-\sigma^2)^n e^{x^2/(2\sigma^2)} \frac{d^n}{dx^n} e^{-x^2/(2\sigma^2)} \quad \sigma^2 > 0,\ m \geq 0.$$

Using the above proposition and the notation in (2.2.7) we obtain the following.

<u>Proposition 3.1.3</u> Let H be a Gaussian space and $h_i \in H$ $i=1,\ldots,n$. Then

$$h_1 \circledcirc \ldots \circledcirc h_n = \tag{3.1.8}$$

$$(n!)^{-3/2}\sum_{\ell=0}^{n-1}(-1)^{\ell}\sum_{N\in P_\ell} h_n\Big(\sum_{i=1}^{n} 1_{N^c}(i)h_i, E\big(\sum_{i=1}^{n} 1_{N^c}(i)h_i\big)^2\Big).$$

The proof follows from (2.2.7) and (3.1.7).

The first few expressions given by (3.1.8) are

$$h_i \circledcirc h_2 = (2)^{-\frac{1}{2}}\{h_1h_2 - E(h_1h_2)\} \tag{3.1.9}$$

$$h_1 \circledcirc h_2 \circledcirc h_3 = (3!)^{-\frac{1}{2}}\{h_1h_2h_3 - h_1E(h_2h_3) - h_2E(h_1h_3) - h_3E(h_1h_2)\} \tag{3.1.10}$$

$$\begin{aligned} h_1 \circledcirc h_2 \circledcirc h_3 \circledcirc h_4 = (4!)^{-\frac{1}{2}}\{&h_1h_2h_3h_4 - h_1h_2E(h_3h_4) - h_1h_3E(h_2h_4) \\ &- h_2h_4E(h_1h_3) - h_2h_3E(h_1h_4) - h_1h_4E(h_2h_3) - h_3h_4E(h_1h_2) \\ &+ E(h_1h_2)E(h_3h_4) + E(h_1h_3)E(h_2h_4) + E(h_1h_4)E(h_2h_3)\} \end{aligned} \tag{3.1.11}$$

where $h_1,\ldots,h_4 \in H$. They are called Multivariate Hermite polynomials (Fox and Taqqu (1984)).

Using the last two propositions we are now able to compute the symmetric tensor product measure for some sets in A^n.

Corollary 3.1.1 Let W_i $i=1,\ldots,n$ be Gaussian random measures as in Proposition 3.1.1. Then if $A_1,\ldots,A_n$ are disjoint sets in A

$$\mathop{\circledcirc}_{i=1}^{n} W_i(A_1\times\ldots\times A_n) = (n!)^{-\frac{1}{2}}\, W_1(A_1)\ldots W_n(A_n). \tag{3.1.12}$$

Proof By (3.1.6) in Proposition 3.1.1

$$\mathop{\circledcirc}_{i=1}^{n} W_i(A_1\times\ldots\times A_n) = W_1(A_1)\circledcirc\ldots\circledcirc W_n(A_n).$$

Since $A_1,\ldots,A_n$ are disjoint sets in A, then by (3.1.2) $W_1(A_1),\ldots,W_n(A_n)$ are orthogonal elements in H where the latter is defined in (3.1.3). Then (3.1.12) follows using Proposition 3.1.2 with $k=n$, $n_i=1$, $h_i=W_i(A_i)$

$i=1,\ldots,n$, and since $h(x;\sigma^2) = x$.

Q.E.D.

Corollary 3.1.2 Let $W_1,\ldots,W_n$ be Gaussian random measures as in Proposition 3.1.1. Then if $A_i \in \mathcal{A}$ $i=1,\ldots,n$

(3.1.13)
$$\bigotimes_{i=1}^{n} W_i(A_1\times\ldots\times A_n)$$

$$(n!)^{-3/2} \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{N\in P_\ell} h_n\Big(\sum_{i=1}^{n} 1_{N^c}(i)W_i(A_i), E\big(\sum_{i=1}^{n} 1_{N^c}(i)W_i(A_i)\big)^2\Big) .$$

The proof follows by (3.1.6) in Proposition 3.1.1 and Proposition 3.1.3.

In the next corollary we consider the special case in which $W = W_1 = \ldots = W_n$ and $\mu_{ij} = \mu$ $i,j=1,\ldots,n$. We use the notation of Proposition 2.2.1.

Corollary 3.1.3 If $A \in \mathcal{A}$

$$W^{\otimes n}(A\times\ldots\times A) = [W(A)]^{\otimes n} = (n!)^{-\frac{1}{2}} h_n(W(A);\mu(A)) .$$

The proof follows from Proposition 3.1.2 since $\mu(A) = E(W(A))^2$ and

$$h^{\otimes n} = (n!)^{-\frac{1}{2}} h_n(h,E(h^2)) .$$

Multiple integrals Let $W_1,\ldots,W_n$ and $\bigotimes_{i=1}^{n} W_i$ be as in Proposition 3.1.1. Using the notation of Section 2.3 we have that if $f\in L_1(\bigotimes_{i=1}^{n} W_i)$, i.e. f is $\bigotimes_{i=1}^{n} W_i$-integrable (see Definition 2.3.1), then

(3.1.14)
$$I_n(f;W_1,\ldots,W_n) = \int_{T^n} f(\underline{t})\, d \bigotimes_{i=1}^{n} W_i(\underline{t})$$

is an element of $H^{\otimes n}$ (and by (3.1.4) an element of $L^2(\Omega,F^W,P)$) which satisfies all properties of the integral w.r.t. the symmetric tensor product measure constructed in Section 2.3. Moreover, we have the next result, in which we use the notation of Theorem 2.3.3.

<u>Lemma 3.1.1</u> Let $W_1,\ldots,W_n$ and $\overset{n}{\underset{i=1}{\otimes}} W_i$ be as in Proposition 3.1.1. Then

a) If $f \in L_1(\overset{n}{\underset{i=1}{\otimes}} W_i)$, $E(I_n(f;W_1,\ldots,W_n)) = 0$.

b) If $f \in L^2(T^n,A^n,\overset{n}{\underset{i=1}{\otimes}} \mu_i)$ and $g \in L^2(T^m,A^m,\overset{m}{\underset{i=1}{\otimes}} \mu_i)$,

$$E(I_n(f;W_1,\ldots,W_n)I_m(g;W_1,\ldots,W_m))$$

$$= \delta_{nm} \int_{T^n} f_{\otimes n}(\underline{t})' \, R^{\otimes n}(\underline{t}) g_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t})$$

where $\mu_o = \sum_{i=1}^{n} \mu_i$, $R^{\otimes n}(\underline{t})$ is defined in (2.3.8) for μ_{ij} $i,j=1,\ldots,n$ as in (3.1.1) and (3.1.2), and $f_{\otimes n}(\underline{t})$ is given by (2.3.11).

<u>Proof</u> a) Since $I_n(f;W_1,\ldots,W_n)$ is $H^{\otimes n}$-valued and $H^{\otimes n}$ is orthogonal to $H^{\otimes 0} \equiv \mathbb{R}$, then $E(I_n(f;W_1,\ldots,W_n)) = 0$.

b) It follows from Theorem 2.3.3 and (3.1.4).

Q.E.D.

Two special cases of the multiple stochastic integral $I_n(f;W_1,\ldots,W_n)$ $= \int_{T^n} f(\underline{t}) d \overset{n}{\underset{i=1}{\otimes}} W_i(\underline{t})$ are now considered. First assume the situation studied by Fox and Taqqu (1984): Let $\mu_{ij}(A) = s_{ij}\mu(A)$ $A \in A$, $i,j=1,\ldots,n$, where μ is a (σ-finite)-measure on (T,A) and $S = (s_{ij})$ in an $n \times n$ non-negative definite matrix, i.e. $W_1,\ldots,W_n$ are Gaussian random measures such that for $i,j=1,\ldots,n$

$$E(W_i(A)W_j(B)) = s_{ij}\mu(A \cap B) \qquad A,B \in A. \tag{3.1.15}$$

Fox and Taqqu following Itô (1951), define the multiple stochastic integral with dependent integrators $J_n(f;W_1,\ldots,W_n)$ in the following manner: Let f be a special elementary function on T^n, i.e.

$$f(\underline{t}) = \sum_{i_1\ldots i_n=1} a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}(\underline{t}) \tag{3.1.16}$$

where $A_1,\ldots,A_p$ is a collection of disjoint sets in $\mathcal{A}$ and $a_{i_1\ldots i_n} \in \mathbb{R}$ are zero unless $i_1,\ldots,i_n$ are all distinct. Denote by $\mathcal{S}_n$ the class of all special elementary functions. If μ satisfies the continuity property (i.e. $\forall\ \epsilon > 0$ and $A \in \mathcal{A}$, $\mu(A) < \infty$, there exist some disjoint $B_j \in \mathcal{A}$, $\mu(B_j) < \varepsilon$ $j=1,\ldots,m$ and $A = \bigcup_{j=1}^{m} B_j$) then $\mathcal{S}_n$ is a dense linear manifold in $L^2(T^n,\mathcal{A}^n,\mu^{\otimes n})$ (Itô (1951)). For $f \in \mathcal{S}_n$ define

$$J_n(f;W_1,\ldots,W_n) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} W_1(A_{i_1})\ldots W(A_{i_n}). \tag{3.1.17}$$

Then J_n can be extended to a bounded linear operator from $L^2(T^n,\mathcal{A}^n,\mu^{\otimes n})$ to $L^2(\Omega,\mathcal{F}^W,P)$ (Fox and Taqqu (1984)).

On the other hand, using Definition 2.3.1 and Corollary 3.1.1 we have that for $f \in \mathcal{S}_n$

$$\begin{aligned} I_n(f;W_1,\ldots,W_n) &= \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} W_1(A_{i_1})\otimes\ldots\otimes W_n(A_{i_n}) \\ &= (n!)^{-\frac{1}{2}} \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} W_1(A_{i_1})\ldots W_n(A_{i_n}) . \end{aligned} \tag{3.1.18}$$

Therefore from (3.1.17) and (3.1.18)

$$I_n(f;W_1,\ldots,W_n) = (n!)^{-\frac{1}{2}} J_n(f;W_1,\ldots,W_n) \quad \text{if } f \in \mathcal{S}_n . \tag{3.1.19}$$

Proposition 3.1.4 Let μ be a finite measure on $(T,\mathcal{A})$ satisfying the continuity property. Then if $f \in L^2(T^n,\mathcal{A}^n,\mu^{\otimes n})$

$$I_n(f;W_1,\ldots,W_n) = (n!)^{-\frac{1}{2}} J_n(f;W_1,\ldots,W_n) .$$

The proof follows since I_n and $(n!)^{-\frac{1}{2}} J_n$ are $L^2(\Omega)$-valued continuous bounded operators on $L^2(T^n,\mathcal{A}^n,\mu^{\otimes n})$ which agree (see (3.1.19)) on the dense linear manifold $\mathcal{S}_n$.

Hence the multiple stochastic integral with dependent integrators of Fox and Taqqu (1984) is a special case of our integral $I_n(f;W_1,\dots,W_n)$ of (3.1.14). Moreover, we do not need to assume the continuity property on μ to construct this integral, although we require μ to be finite (see beginning of Section 2.1).

Next assume the situation considered by Itô (1951): $W = W_1,\dots,W_n$ is a Gaussian random measure on (T,A) and μ satisfies the continuity property. Itô's multiple Wiener integral $J_n(f;W)$ is constructed as in the Fox and Taqqu case above and it is a bounded linear operator from $L^2(T^n,A^n,\mu^{\otimes n})$ to $L^2(\Omega,F^W,P)$. As in Proposition 3.1.4 we have that for $f \in L^2(T^n,A^n,\mu^{\otimes n})$ $I_{n\odot}(f;W) = (n!)^{-\frac{1}{2}} J_n(f,W)$ where $I_{n\odot}$ is as in Proposition 2.3.6.

The next two propositions were obtained by Itô (1951). They relate multiple Wiener integrals with Hermite polynomials. We prove them here using the symmetric tensor product set up. We remark that since in the construction of $I_{n\odot}(f;W)$ in Section 2.3 it was not required that μ satisfies the continuity property (as it is required in Itô's case) this will not be assumed in the next result.

Proposition 3.1.5 Let $\phi_1(t),\dots,\phi_m(t)$ be an orthogonal system of real valued functions in $L^2(T,A,\mu)$ and h_k be the Hermite polynomial of degree k given in Proposition 3.1.2. Define

$$f(\underline{t}) = \phi_1(t_1)\dots\phi_1(t_{p_1})\phi_2(t_{p_1+1})\dots\phi_2(t_{p_1+p_2})\dots\phi_m(t_{p_1+\dots+p_{m-1}})\dots\phi_m(t_{p_1+\dots+p_m}).$$

Then if $n = p_1+\dots+p_m$

$$I_{n\odot}(f) = (n!)^{-\frac{1}{2}} \prod_{i=1}^{m} h_{p_i}(I_1(\phi_i);\ E(I_1(\phi_i))^2)$$

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where $I_1(g) = \int_T g(s)dW(s)$ is the isometric integral w.r.t. W given in Theorem 2.1.1.

Proof From Lemma 2.3.1 f is $W^{\circledast n}$-integrable and

$$I_{n\circledast}(f;W) = I_1(\phi_1)^{\circledast p_1}\circledast\ldots\circledast I_1(\phi_m)^{\circledast p_m} .$$

Next since for $i\neq j$ $E(I_1(\phi_i)I_1(\phi_j)) = \int_T \phi_i(s)\phi_j(s)d\mu(s) = 0$ then by Proposition 3.1.2

$$I_1(\phi_1)^{\circledast p_1}\circledast\ldots\circledast I_1(\phi_m)^{\circledast p_m} = (n!)^{-\frac{1}{2}} \prod_{j=1}^{m} h_{p_j}(I_1(\phi_j);\ E(I_1(\phi_j))^2).$$

Q.E.D.

Proposition 3.1.6 Let $T = [0,1]$ and μ be non-atomic. Then if

$$T_1^n = \{(t_1,\ldots,t_n)\in T^n:\ 0\le t_1<\ldots< t_n\le 1\}$$

$$I_{n\circledast}(1_{T_1^n};W) = (n!)^{-3/2} h_n(W(T);\mu(T)) \ .$$

That is formally

$$(3.1.20)\qquad \int_{0\le t_1<\ldots<t_n\le 1}\ldots\int dW(t_1)\ldots dW(t_n) = (n!)^{-3/2} h_n(W(T);\mu(T)).$$

Proof By Definition 2.3.1 $I_{n\circledast}(1_{T_1^n};W) = W^{\circledast n}(T_1^n)$. Then the result follows by Corollary 3.1.3, since $W^{\circledast n}$ is finitely additive, μ is non-atomic and Proposition 2.2.1 (c).

Q.E.D.

A different proof of (3.1.20) above is given in Theorem 6.5 of Engel (1982).

3.2 Poisson randon measure

Let (Ω,F,P) be a complete probability space and (T,A) be any measurable space such that all singleton sets are measurable, i.e. $\{t\}\in A\ \ \forall\ t\in T$.

Let $M(T)$ be the set of all σ-finite measures on (T,A). We assume that N is a Poisson random measure on (T,A) with intensity $\mu \in M(T)$, i.e. $N(A)$ $A \in A$ is an integer-valued random measure such that the following two conditions hold:

(i) for each $A \in A$, $\mu(A) < \infty$, $N(A)$ is a Poisson random variable with mean $\mu(A)$,

(ii) if $A_1,\ldots,A_k$ are disjoint sets in A then $N(A_1),\ldots,N(A_k)$ are independent random variables on (Ω,F,P).

The signed random measure $q(A) = N(A) - \mu(A)$ $A \in A$, $\mu(A) < \infty$ is called a centered Poisson random measure. Since for $A,B \in A$ with $\mu(A) < \infty$, $\mu(B) < \infty$, $E(q(A)q(B)) = \mu(A \cap B)$, then q is an orthogonally scattered measure on (T,A) with control measure μ. We assume μ is a non-atomic measure.

In this section we consider symmetric tensor product measures of the centered Poisson random measure q with itself for all $n \geq 1$. That is, using the notation of Proposition 2.2.1 we construct $q^{\otimes n}$ for $n \geq 1$ and multiple integrals w.r.t. $q^{\otimes n}$.

Let $F^q = \sigma(N(A): A \in A, \mu(A) < \infty)$, $I_q(f)$ be the isometric integral of f w.r.t. q (Theorem 2.1.1) for $f \in L^2(T,A,\mu)$, and H_q be the subspace of $L^2(\Omega,F^q,P)$ generated by q (see (2.1.3)), i.e.

$$H_q = \{I_q(f): f \in L^2(T,A,\mu)\}.$$

The exponential space $EXP(H_q)$ associated with a Poisson random measure has been studied by Neveu (1968) and Surgailis (1984) for the cases when the control measure μ is finite and σ-finite respectively. We do not follow the identification of $EXP(H_q)$ given by the second named author since he uses multiple Poisson integrals to obtain this identification and we want to proceed in the opposite way: first identify $EXP(H_q)$, then obtain

the symmetric tensor product stochastic measure $q^{\circledast n}$ and finally construct multiple integrals w.r.t. $q^{\circledast n}$, as we did in the last section for the Gaussian random measure. Although for the purpose of this section the identification of $\mathrm{EXP}(H_q)$ given by Neveu (1968) (μ finite) is sufficient, in the next theorem we extend Neveu's result to the case when μ is σ-finite. This result will be used in Section 3.3 where we study the general L^2-independent increments processes case.

Theorem 3.2.1 Let (Ω, F, P) be a probability space in which there is defined a centered Poisson random measure q on a measurable space (T, A) with σ-finite non-atomic control measure μ. Then

$$\mathrm{EXP}(H_q) \overset{\phi}{\cong} L^2(\Omega, F^q, P)$$

where for $f \in L^2(T, A, \mu)$

$$\phi(\exp \circledast (I_q(f)) = \left\{ \prod_{i=1}^{\infty} \prod_{j=1}^{N(T_i)} (1+f(Z_j^{(i)}))e^{-\int_{T_i} f d\mu} \right\} \tag{3.2.1}$$

where

(i) T_i $i \geq 1$ are disjoint sets in A, $0 < \mu(T_i) < \infty$ and $\bigcup_{i=1}^{\infty} T_i = T$,

(ii) for each $i=1,2,\ldots$ and $j=1,2,\ldots$, $Z_j^{(i)}$ is a T_i-valued random element with distribution given by the measure $\mu(T_i)^{-1}\mu(\cdot)$, and for each $i=1,2,\ldots$ $N(T_i)$ follows a Poisson distribution with parameter $\mu(T_i)$,

(iii) $Z_j^{(i)}, N(T_i)$ $i=1,2,\ldots,$ $j=1,2,\ldots$ are mutually independent.

Moreover,

$$E(\phi(\exp \circledast (I_q(f)))^2 = \exp(E(I_q(f))^2) = \exp(\int_T f^2 d\mu) < \infty.$$

In order to prove the theorem we first prove the following technical result.

Lemma 3.2.1 Let μ and $T_i, N(T_i)$, $Z_j^{(i)}$ $j=1,2,\ldots$, $i=1,2,\ldots$ be as in (i)-(iii) in the above theorem. If $g \in L^1(T_i, A\cap T_i, \mu)$ for some $i \geq 1$ then

$$E\left(\prod_{j=1}^{N(T_i)} g(Z_j^{(i)})\right) = e^{\int_{T_i}(g-1)d\mu} .$$

Proof Since $N(T_i)$ follows the Poisson distribution with parameter $\mu(T_i) < \infty$ and for each $j=1,2,\ldots$ $Z_j^{(i)}$ has distribution $\mu(T_i)^{-1}\mu(\cdot)$, then using the independence of $N(T_i)$, $Z_1^{(i)}$, $Z_2^{(i)}$, ...

$$E\left(\prod_{j=1}^{N(T_i)} g(Z_j^{(i)})\right) = \sum_{n=0}^{\infty} \frac{e^{-\mu(T_i)}}{n!} \int_{T_i^n} \prod_{j=1}^{n} g(t_j) \prod_{j=1}^{n} d\mu(t_j)$$

$$= e^{-\mu(T_i)} e^{\int_{T_i} g d\mu} = e^{\int_{T_i}(g-1)d\mu} .$$

Q.E.D.

Proof of Theorem 3.2.1 For any Hilbert space K $\{\exp \circledast(k) : k \in K\}$ generates $EXP(K)$ (Guichardet (1972)), where

$$\exp \circledast(k) = (1, k, (2!)^{-\frac{1}{2}} k^{\circledast 2}, \ldots)$$

and

$$\langle \exp \circledast(k_1), \exp \circledast(k_2) \rangle_{EXP(K)} = e^{\langle k_1, k_2 \rangle_K} \qquad k_1, k_2 \in K.$$

Then since $L^2(T,A,\mu) \overset{I_q}{\cong} H_q$, in order to prove the theorem we have to show the following three conditions:

a) for each $f \in L^2(T,A,\mu)$ $\quad \phi(\exp \circledast(I_q(f))) \in L^2(\Omega, F^q, P)$.

b) for $f_1, f_2 \in L^2(T,A,\mu)$ $\quad E(\phi(\exp \circledast(I_q(f_1))\phi(\exp \circledast(I_q(f_2)))$

$$= \langle \exp \circledast(I_q(f_1)), \exp \circledast(I_q(f_2)) \rangle_{EXP(H_q)} = e^{\int_T f_1 f_2 d\mu} .$$

c) $\{\phi(\exp \circledast(I_q(f))) : f \in L^2(T,A,\mu)\}$ generates $L^2(\Omega, F^q, P)$.

Since μ is a σ-finite measure on (T,A), there exists a sequence of sets $\{T_i\}_{i\geq 1}$ in A such that $0<\mu(T_i)<\infty$ and $\bigcup_{i=1}^{\infty} T_i = T$. The existence of the random elements $Z_j^{(i)}$ $j=1,2,\ldots,$ $i=1,2,\ldots$ satisfying (ii) and (iii) follows from the construction of a Poisson random measure N with control measure μ (Theorem 8.1 in Ikeda and Watanabe (1981)).

Let $f\in L^2(T,A,\mu)$, then for each $i\geq 1$ f belongs to $L^2(T_i,A\cap T_i,\mu)$ and $L^1(T_i,A\cap T_i,\mu)$. Then by taking $g=(1+f)$ in Lemma 3.2.1 we obtain

$$E\left(\prod_{j=1}^{N(T_i)}(1+f(Z_j^{(i)}))\, e^{-\int_{T_i} f d\mu}\right) = 1 \quad i=1,2,\ldots\ .$$

Then using (iii) $\quad G_i = \prod_{j=1}^{N(T_i)}(1+f(Z_j^{(i)}))e^{-\int_{T_i} f d\mu}$ is a sequence of independent random variables with $E(G_i)=1$ $i\geq 1$ and therefore

$$D_n = \prod_{i=1}^{n} G_i$$

is a martingale. Next, using Lemma 3.2.1 with $g=(1+f)^2$ and the independence of $Z_j^{(i)}$, $N(T_i)$ $j=1,2,\ldots,$ $i=1,2,\ldots$

$$E\,D_n^2 = \prod_{i=1}^{n} E\left(\prod_{j=1}^{N(T_i)}(1+f(Z_j^{(i)}))e^{-\int_{T_i} d\mu}\right)^2$$

$$= \prod_{i=1}^{n} e^{-2\int_{T_i} f d\mu}\, E\left(\prod_{j=1}^{N(T_i)}(1+f(Z_j^{(i)}))^2\right) = \prod_{i=1}^{n} e^{-2\int_{T_i} f d\mu}\, e^{\int_{T_i}((1+f)^2-1)d\mu}$$

$$= \prod_{i=1}^{n} e^{\int_{T_i} f^2 d\mu} = \exp\Big(\int_{\bigcup_{i=1}^{n} T_i} f^2 d\mu\Big) \leq \exp\Big(\int_T f^2 d\mu\Big) < \infty$$

i.e.

$$E\,D_n^2 \leq \exp\Big(\int_T f^2 d\mu\Big) < \infty \qquad \text{all } n\geq 1.$$

Then by the martingale convergence theorem D_n converges a.s. and in mean square to $\phi(\exp \circledcirc(I_q(f)))$. Therefore

$$E\left(\prod_{i=1}^{\infty}\prod_{j=1}^{N(T_i)}(1+f(Z_j^{(i)}))e^{-\int_{T_i} f d\mu}\right) = E\, D_n = 1$$

and

$$E\left(\prod_{i=1}^{\infty}\prod_{j=1}^{N(T_i)}(1+f(Z_k^{(i)}))e^{-\int_{T_i} f d\mu}\right)^2 = \lim_{n\to\infty} E\, D_n^2$$

$$= \lim_{n\to\infty} \exp\left(\int_{\bigcup_{i=1}^{n} T_i} f^2 d\mu\right) = \exp\left(\int_T f^2 d\mu\right) < \infty$$

which shows (a).

Let $f_1, f_2 \in L^2(T,A,\mu)$, then applying Lemma 3.3.1 to $g = (1+f_1)(1+f_2)$ one shows in a similar way as above that

$$E(\exp \circledast(I_q(f_1)) \exp \circledast(I_q(f_2))) = \lim_{n\to\infty} \prod_{i=1}^{n} E\left(\prod_{j=1}^{N(T_i)}(1+f_1)(1+f_2)(Z_j^{(i)})e^{-\int_{T_i}(f_1^2+f_2^2)d\mu}\right)$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} e^{\int_{T_i} f_1 f_2 d\mu} = e^{\int_T f_1 f_2 d\mu}$$

proving (b).

Finally, to prove (c) let $G \in L^2(\Omega, F^q, P)$ and suppose that

$$E(\exp \circledast(I_q(f))G) = 0 \quad \text{for all } f \in L^2(T,A,\mu).$$

We want to show that $G = 0$ a.e. dP_{F^q}, where

$$F^q = \sigma(I_q(f) : f \in L^2(T,A,\mu)).$$

Using (3.2.1) we have that for all $f \in L^2(T,A,\mu)$

$$E\left(\prod_{i=1}^{\infty}\left\{\prod_{j=1}^{N(t_i)}(1+f(Z_j^{(i)}))e^{-\int_{T_i} f d\mu}\right\} G\right) = 0.$$

Next let $i \geq 1$ be fixed and for $g \in L^2(T_i, A \cap T_i, \mu)$ define $f: T \to \mathbb{R}$ by $f(t) = g(t)$ $t \in T_i$ and zero if $t \notin T_i$. Then $f \in L^2(T,A,\mu)$ and

$$E\left(\prod_{j=1}^{N(T_i)} (1+g(Z_j^{(i)}))e^{-\int_{T_i} g d\mu} G \right) = 0 \quad \text{all } g \in L^2(T_i A \cap T_i, \mu).$$

Hence by Proposition 7.13 in Neveu (1968) (Lemma 3.3.2 below) $E(G|F_i^q) = 0$ a.s. where $F_i^q = \sigma(I_q(g): g \in L^2(T_i, A \cap T_i, \mu))$, and $F_i^q \subset F^q$ all $i \geq 1$.

Hence for all $n \geq 1$ $E(G| \bigvee_{i=1}^{n} F_i^q) = 0$ a.s. since $F_1^q, \ldots, F_n^q$ are independent σ-fields.

Next let $F_n = \bigvee_{i=1}^{n} F_i^q$ and $F_\infty = \bigvee_{n=1}^{\infty} F_n \subset F^q$. Then since $E(G^2) < \infty$ by the martingale convergence theorem $G = 0$ a.s. dP_{F^∞}. Thus is remains to show that $F^q \subset F^\infty$.

Let $f \in L^2(T, A, \mu)$, then

$$f(t) = \sum_{i=1}^{\infty} f(t) 1_{T_i}(t)$$

and

$$I_q(f) = \sum_{i=1}^{\infty} I_q(f \cdot 1_{T_i}) \qquad \text{a.s. .}$$

Thus $I_q(f)$ is F^∞-measurable all $f \in L^2(T, A, \mu)$ since for each $i \geq 1$ $I_q(f \cdot 1_{T_i})$ is F_i^q-measurable. That is, $F^q \subset F^\infty$ and $G = 0$ a.e. dP_{F^q}.

Q.E.D.

We are now going to apply our results of Chapter 2 to the construction of the product stochastic measure $q^{\otimes n}$. Therefore in the remainder of this section we will assume that the control measure μ is finite and the following version (Proposition 7.13 in Neveu (1968)) of the last theorem will be enough for our purpose.

Lemma 3.2.2 (Neveu (1968)). Let q be a centered Poisson random measure as in Theorem 3.2.1 with finite control measure μ. Then

$$\mathrm{EXP}(H_q) \overset{\eta}{\cong} L^2(\Omega, F^q, P)$$

where for $f \in L^2(T,A,\mu)$

$$(3.2.2) \qquad \eta(\exp \circledast(I_q(f)) = \prod_{j=1}^{N(T)} (1+f(Z_j))e^{-\int_T f d\mu}$$

where $\{Z_j\}_{j\geq 1}$ is a sequence of independent random elements, independent of $N(T)$, each Z_j taking values in T and having distribution $\{\mu(T)\}^{-1}\mu(\cdot)$.

Using the identification of $EXP(H_q)$ given by the above lemma, in our next result we obtain an L^2-valued product stochastic measure of the Poisson random measure q.

Proposition 3.2.1 Let q be a centered Poisson random measure on (T,A) with finite non-atomic control measure μ. Then for each $n\geq 1$ there exists a unique $L^2(\Omega,F^q,P)$-valued measure $q^{\circledast n}$ on (T^n,A^n) such that if $A_1,\ldots,A_n$ belong to A

$$(3.2.3) \qquad q^{\circledast n}(A_1\times\ldots\times A_n) = q(A_1)\circledast\ldots\circledast q(A_n)$$

and for $A \in A^n$

$$E(q^{\circledast n}(A)) = 0$$

$$VAR(q^{\circledast n}(A)) = \frac{1}{n!} \sum_{\Pi} \mu^{\circledast n}(A\cap A^{\Pi}).$$

Moreover, Proposition 2.2.1 (b)-(d), Lemma 2.2.6 and Corollaries 2.2.8-2.2.9 hold for $q^{\circledast n}$.

Proof For each $n\geq 1$, existence and uniqueness of the $H_q^{\circledast n}$-valued measure $q^{\circledast n}$ follow from Proposition 2.2.1 (a). It is seen as an $L^2(\Omega,F^q,P)$-valued element by Lemma (3.2.2). The expressions for the mean and the variance follow similar to Proposition 3.1.1.

Q.E.D.

We now compute special cases of (3.2.3). If $A\in A$ then $q(A) = I_q(1_A)$ and from (3.2.2) we have that

$$q(A)^{\circledast n} = (n!)^{-\frac{1}{2}} \left(\frac{d^n}{dz^n}\right)_{z=0} \exp \circledast(zq(A))$$

$$= (n!)^{-\frac{1}{2}} \left(\frac{d^n}{dz^n}\right)_{z=0} \prod_{i=1}^{N(T)} (1+z1_A(Z_i))e^{-z\int_T 1_A d\mu}$$

$$= (n!)^{-\frac{1}{2}} \left(\frac{d^n}{dz^n}\right)_{z=0} (1+z)^{N(A)} e^{-z\mu(A)} .$$

But $(1+z)^x e^{-z\lambda}$ $z > -1$ is the generating function of the Poisson-Charlier polynomials (Chihara (1978)) with parameter $\lambda > 0$, denoted by $c_n(x;\lambda)$, i.e.

(3.2.4) $$e^{-\lambda z}(1+z)^x = \sum_{n=0}^{\infty} \frac{z^n}{n!} c_n(x;\lambda) \qquad \lambda > 0.$$

Then for $A \in \mathcal{A}^n$

(3.2.5) $$q^{\circledast n}(A) = (n!)^{-\frac{1}{2}} c_n(N(A);\mu(A)).$$

We now obtain a similar result to Proposition 3.1.2 for symmetric tensor products of Poisson random measures.

<u>Proposition 3.2.2</u> Let q be a centered Poisson random measure as in Proposition 3.2.1 and $A_1,\ldots,A_k$ disjoint sets in $\mathcal{A}$. Then

(3.2.6) $$q(A_1)^{\circledast n_1}\circledast\ldots\circledast q(A_k)^{\circledast n_k} = (n!)^{-\frac{1}{2}} \prod_{j=1}^{k} c_{n_j}(N(A_j);\ \mu(A_j))$$

where $n = \sum_{j=1}^{k} n_j$ and $c_m(x;\lambda)$ are Poisson-Charlier polynomials with parameter λ defined in (3.2.4).

<u>Proof</u> Since $A_1,\ldots,A_k$ are disjoint sets in $\mathcal{A}$, then $q(A_1),\ldots,q(A_n)$ are mutually orthogonal in $H_q \subset L^2(\Omega,\mathcal{F}^q,P)$ and therefore the family $\{q(A_1)^{\circledast n_1}\circledast\ldots\circledast q(A_k)^{\circledast n_k};\ n_1 \geq 0,\ldots,n_k \geq 0\}$ is orthogonal in $\mathrm{EXP}(H_q)$. Next for $z_1,\ldots,z_k \in \mathbb{R}$

$$\exp \circledast(z_1q(A_1)+\ldots+z_kq(A_k)) = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}}(z_1q(A_1)+\ldots+z_kq(A_k))^{\circledast n}$$

$$= \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \sum_{n_1+\dots+n_k=n} \frac{z_1^{n_1}}{n_1!} \cdots \frac{z_k^{n_k}}{n_k!} q(A_1)^{\otimes n_1} \otimes \dots \otimes q(A_k)^{\otimes n_k}$$

$$= \sum_{n_1 \dots n_k} \frac{((n_1+\dots+n_k)!)^{-\frac{1}{2}}}{n_1! \dots n_k!} z_1^{n_1} \dots z_k^{n_k} q(A_1)^{\otimes n_1} \otimes \dots \otimes q(A_k)^{\otimes n_k} .$$

On the other hand from (3.2.2)

$$\eta(\exp \otimes (z_1 q(A_1) + \dots + z_k q(A_k))$$

$$= \prod_{i=1}^{N(T)} (1 + z_1 1_{A_1}(Z_i) + \dots + z_k 1_{A_k}(Z_i)) \, e^{-(z_1 \mu(A_1) + \dots + z_k \mu(A_k))}$$

$$= \prod_{i=1}^{k} (1+z_i)^{N(A_i)} e^{-z_i \mu(A_i)} .$$

Then the assertion of the proposition follows from the last two expressions since

$$q(A_1)^{\otimes n_1} \otimes \dots \otimes q(A_k)^{\otimes n_k} = (n!)^{-\frac{1}{2}} \left(\frac{\partial^{n_1+\dots+n_k}}{\partial z_1^{n_1} \dots \partial z_k^{n_k}} \right)_{\underline{z}=\underline{0}} \prod_{i=1}^{k} (1+z_i)^{N(A_i)} e^{-z\mu(A_i)}$$

$$(\underline{z} = (z_1, \dots, z_k)) .$$

Q.E.D.

Expression (3.2.6) is related to the <u>multivariate Poisson-Charlier polynomials</u> (Ogura (1972), Chihara (1978)) defined by

$$K_n((x_1)_{n_1}, \dots, (x_k)_{n_k}; (\lambda_1)_{n_1}, \dots, (\lambda_k)_{n_k}) =$$

$$\left(\frac{\partial^n}{\partial z_1^{n_1} \dots \partial z_k^{n_k}} \right)_{\underline{z=0}} \prod_{i=1}^{k} (1+z_i)^{x_i} e^{-\lambda_i z_i}$$

where $n = \sum_{i=1}^{k} n_i$, $(x_i)_{n_i} = (\underbrace{x_i, \dots, x_i}_{n_i \text{ times}})$ and $\lambda_i > 0 \quad i=1,\dots,k$.

The first few expressions are

$$K_0(x;\lambda) = 1, \quad K_1(x;\lambda) = c_1(x;\lambda) = x-\lambda$$

$$K_2(x_1,x_2;\lambda_1,\lambda_2) = x_1(x_2-\delta_{12}) - x_1\lambda_1 - x_2\lambda_2 + \lambda_1\lambda_2$$

$$K_3(x_1,x_2,x_3;\lambda_1,\lambda_2,\lambda_3) = x_1(x_2-\delta_{12})(x_3-\delta_{13}-\delta_{23})$$

$$- [x_1(x_2-\delta_{12})\lambda_1\lambda_2 + x_2(x_3-\delta_{23})\lambda_2\lambda_3 + x_3(x_1-\delta_{13})\lambda_1\lambda_3]$$

$$+ x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 - \lambda_1\lambda_2\lambda_3$$

where $\delta_{ij} = 1$ if $i=j$, and 0 otherwise.

Corollary 3.2.1 If $A_1,\dots,A_n$ are disjoint sets in $\mathcal{A}$

$$q(A_1)\circledast\dots\circledast q(A_n) = (n!)^{-\frac{1}{2}} q(A_1)\dots q(A_n).$$

Proof Taking $k=n$, $n_i=1$ $i=1,\dots,n$ in Proposition 3.2.2 since $q(A_1),\dots,q(A_n)$ are orthogonal

$$q(A_1)\circledast\dots\circledast q(A_n) = (n!)^{-\frac{1}{2}} \prod_{i=1}^{n} c_1(N(A_i);\ \mu(A_i)).$$

Then the corollary follows since $c_1(N(A);\ \mu(A)) = N(A) - \mu(A) = q(A)$ for all $A \in \mathcal{A}$.

Q.E.D.

Multiple Poisson integrals Using the notation of Section 2.3., if $f \in L_1(q^{\circledast n})$ $n \geq 1$ then

$$I_{n\circledast}(f;q) = \int_{T^n} f(\underline{t})dq^{\circledast n}(\underline{t})$$

is an element of $H_q^{\circledast n}$ (and hence of $L^2(\Omega,\mathcal{F}^q,P)$ by (3.2.1)). This integral satisfies all properties of the integral w.r.t. the symmetric tensor product measure of Section 2.3, and in particular Propositions 2.3.6 and 2.3.7

now written in the following manner.

Lemma 3.2.3 a) If $f \in L_1(q^{\otimes n})$, $E(I_{n\otimes}(f;q)) = 0$.

b) If $f \in L^2(T^n, A^n, \mu^{\otimes n})$ and $g \in L^2(T^m, A^m, \mu^{\otimes m})$

$$E(I_{n\otimes}(f;q) I_{m\otimes}(g;q)) = \delta_{nm} \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^n, A^n, \mu^{\otimes n})} .$$

c) $\{I_{n\otimes}(f;q) : f \in L^2(T^n, A^n, \mu^{\otimes n})\ n \geq 0\}$ constitutes a complete orthogonal system in $L^2(\Omega, F^q, P)$, where $I_{0\otimes}(\cdot) = 1$.

Proof a) follows from Proposition 3.2.1, (b) from Proposition 2.3.6 and (c) by (3.2.1) and Proposition 2.3.7.

Q.E.D.

Multiple Poisson Wiener integrals were introduced by Itô (1956) and studied later by Ogura (1972) and Surgailis (1984). H. Ogura uses multivariate Poisson-Charlier polynomials to define a multiple stochastic integral with respect to q (when $T \subseteq \mathbb{R}$), which we denote by $J_n(f)$. If f is an elementary function defined in (2.3.17)

$$J_n(f) = \sum_{i_1 \ldots i_n = 1}^{p} a_{i_1 \ldots i_n} K_n(N(A_{i_1}), \ldots, N(A_{i_n}); \mu(A_{i_1}), \ldots, \mu(A_{i_n}))$$

where K_n is multivariate Poisson-Charlier polynomial. Then J_n extends to a bounded linear operator from $L^2(T^n, A^n, \mu^{\otimes n})$ to $L^2(\Omega, F^q, P)$ (Ogura (1972)).

On the other hand, using Proposition 3.2.2, if f is an elementary function

$$I_{n\otimes}(f;q) = \sum_{i_1 \ldots i_n = 1}^{p} a_{i_1 \ldots i_n} q(A_{i_1}) \otimes \ldots \otimes q(A_{i_n})$$

$$= (n!)^{-\frac{1}{2}} \sum_{i_1 \ldots i_n = 1}^{p} a_{i_1 \ldots i_n} K_n(N(A_{i_1}), \ldots, N(A_{i_n}); \mu(A_{i_1}), \ldots, \mu(A_{i_n})).$$

Thus $I_{n\otimes}(f;q)$ and $(n!)^{-\frac{1}{2}} J_n(f)$ agree on a dense linear manifold of

$L^2(T^n, A^n, \mu^{\otimes n})$ and therefore the following result is obtained.

Proposition 3.2.3 If $f \in L^2(T^n, A^n, \mu^{\otimes n})$

$$I_{n\otimes}(f;q) = (n!)^{-\frac{1}{2}} J_n(f).$$

Proposition 3.2.4 Let $T = [0,1]$ and μ be non-atomic. Then if $T_1^n = \{(t_1, \ldots, t_n) \in T^n\colon 0 \le t_1 < \ldots < t_n \le 1\}$

$$I_{n\otimes}(1_{T_1^n};q) = (n!)^{-3/2} c_n(N(T);\, \mu(T)).$$

That is, formally

$$\int_{0 \le t_1 < \ldots < t_n \le 1} \!\!\cdots\!\! \int dq(t_1)\ldots dq(t_n) = (n!)^{-3/2} c_n(N(T);\, \mu(T)) \tag{3.2.7}$$

where c_n is the Poisson-Charlier polynomial of degree n defined in (3.2.4).

Proof By definition $I_{n\otimes}(1_{T_1^n};q) = q^{\otimes n}(T_1^n)$. The rest of the proof follows similar to Proposition 3.1.6 using Corollary 3.2.1

The expression (3.2.7) above is Theorem 6.9 in Engel (1982), who gives a different proof.

3.3 Nonidentically distributed L^2-independently scattered measures

In this section we study L^2-valued product stochastic measures and multiple stochastic integrals of nonidentically distributed L^2-independently scattered measures that are mutually independent over disjoint sets. The identification of the exponential space of H, the common Hilbert space where the i.s.m.'s take values, is obtained using results given in the last two sections. The parameter set T considered in this section is an interval of the real line. This is an important assumption throughout the section since we use martingale theory to identify symmetric tensor products of H.

We conclude this section by giving characterizations in terms of multiple stochastic integrals of Poisson and Gaussian processes with independent increments.

Our first result of this section is a general one, in the sense that it identifies the exponential space of any Hilbert space H which is a direct sum of an arbitrary Gaussian space H_W and an arbitrary Poisson space H_q, where H_W and H_q are stochastically independent.

Theorem 3.3.1 Let (Ω, F, P) be a complete probability space and q be a centered Poisson random measure on a measurable space $(E, \mathcal{E})$ defined on (Ω, F, P), with σ-finite non-atomic control measure μ and generating the Poisson space

$$H_q = \{I_q(f) : f \in L^2(E, \mathcal{E}, \mu)\}$$

where I_q is the isometric integral of f w.r.t. q. Let H_W be a Gaussian space on (Ω, F, P) stochastically independent of the system of random variables H_q. Define the σ-fields $F^W = \sigma(H_W)$, $F^q = \sigma(H_q)$ and the Hilbert space

$$H = H_W \oplus H_q \ .$$

Then

$$\text{(3.3.1)} \qquad \mathrm{EXP}(H) = \sum_{n \geq 0} \oplus H^{\odot n} \overset{\gamma}{\cong} L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$$

where for $h \in H$, $h = h_W + h_q$ $h_W \in H_W$, $h_q \in H_q$

$$\gamma: \mathrm{EXP}(H) \to L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$$

is defined by

$$\text{(3.3.2)} \qquad \gamma(\exp \odot(h)) = \psi(\exp \odot(h_W))\, \phi(\exp \odot(h_q))$$

and ψ, ϕ are the isometrics given in (3.1.4) and Theorem 3.2.1 respectively.

<u>Proof</u> We first prove that for all $h \in H$, $\gamma(\exp \circledcirc(h))$ is an element of $L^2(\Omega, F^W \vee F^q, P)$ and that

$$(3.3.3) \qquad E(\gamma(\exp \circledcirc(h)))^2 = \exp(Eh^2).$$

By Theorem 3.2.1 for all $h_q \in H_q$

$$E(\phi(\exp \circledcirc(h_q)))^2 = \exp(Eh_q^2) < \infty$$

and by Proposition 7.3 in Neveu (1968) for all $h_W \in H_W$

$$E(\psi(\exp \circledcirc(h_W)))^2 = \exp(Eh_W^2) < \infty .$$

Then if $h = h_W + h_q$ $\gamma(\exp \circledcirc(h)) = \psi(\exp \circledcirc(h_W))\phi(\exp \circledcirc(h_q))$ belongs to $L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$ since h_W and h_q are independent. Moreover, from the above expressions we have that

$$E(\gamma(\exp \circledcirc(h)))^2 = \exp(Eh_W^2 + Eh_q^2) = \exp(Eh^2).$$

Next we shall prove that $\{\gamma(\exp \circledcirc(h)) : h \in H\}$ generates $L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$, which will imply (3.3.1) since for any Hilbert space K $\{\exp \circledcirc(k) : k \in K\}$ generates the Hilbert space EXP(K) (Guichardet (1972)).

Let $Z \in L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$ and suppose that

$$E(Z\gamma(\exp \circledcirc(h))) = 0 \quad \text{for each } h \in H.$$

Then for all $h_W \in H_W$ and $h_q \in H_q$

$$E(Z\psi(\exp \circledcirc(h_W))\phi(\exp \circledcirc(h_q))) = 0.$$

But from Proposition 7.3 in Neveu (1968) and Theorem 3.2.1 in this thesis $\{\psi(\exp \circledcirc(h_W)) : h_W \in H_W\}$ and $\{\phi(\exp \circledcirc(h_q)) : h_q \in H_q\}$ generate $L^2_{\mathbb{R}}(\Omega, F^W, P)$ and $L^2_{\mathbb{R}}(\Omega, F^q, P)$ respectively. Then for all $A_1 \in F^W$ and $A_2 \in F^q$

$$\int_{A_1 \cap A_2} Z \, dP = 0 .$$

But it is known (Dellacherie and Meyer (1978)) that if two σ-fields F_1 and F_2 are independent, then $F_1 \vee F_2$ is generated by the field C_0 of all finite disjoint unions of sets $A_1 \cap A_2$ $A_1 \in F_1$, $A_2 \in F_2$.

Thus since Z is P-integrable

$$C = \{A \in F: \int_A Z dP = 0\}$$

is a monotone class, and by the monotone class theorem

$$\int_A Z dP = 0 \quad \forall \; A \in F^W \vee F^q$$

since $C_0 \subset C$. That is, $Z = 0$ a.e. $dP_{F^W \vee F^q}$.
Then

$$\{\gamma(\exp \circledcirc(h) : h \in H\}$$

generates the space $L^2_{\mathbb{R}}(\Omega, F^W \vee F^q, P)$.

Q.E.D.

Assumption 3.3.1 Throughout the remainder of this section we will make the following assumptions and notations: Let (Ω, F, P) be a complete probability space, $T = [0, T_o]$ $T_o > 0$, $A = B(T)$, $\mathbb{R}_n^o = \mathbb{R}_n - \{\underline{0}\}$ and $B_n^o = B(\mathbb{R}_n^o)$ for $n \geq 1$. Suppose that

$$(3.3.4) \qquad Y_t = (X_1(t), \ldots, X_n(t)) \quad t \in T, \quad Y_0 = \underline{0}$$

is an n-dimensional stochastically continuous, right continuous, zero mean, L^2-stochastic process with independent increments defined on (Ω, F, P), with characteristic function given by

$$(3.3.5) \qquad \Phi_{Y_t}(\underline{a}) = \exp\{-\tfrac{1}{2} \underline{a}'\sigma(t)\underline{a} + \int_{\mathbb{R}_n^o} (e^{i\underline{a}'\underline{x}} - 1 - i\underline{a}'\underline{x}) \nu(t, d\underline{x})\} \qquad \underline{a} \in \mathbb{R}_n$$

and Levy-Itô representation (Gikhman-Skorokhod (1969))

(3.3.6) $$Y_t = W_t + \int_0^t \int_{\mathbb{R}_n^o} \underline{x}q(ds,d\underline{x}) \qquad \text{all } t \in T$$

where

(3.3.7) W_t is an n-dimensional Gaussian process with independent increments, $W_0 = 0$ and positive definite diffusion matrix $\sigma(t)$;

(3.3.8) $q(B,\Gamma) = N(B,\Gamma) - \nu(B,\Gamma) \qquad \Gamma \in B_n^o,\ \ B \in A$

is a centered Poisson random measure on $(T\times\mathbb{R}_n^o,\ A\times B_n^o)$ with σ-finite control measure ν, and independent of $\{W_t\}$ $t \in T$;

(3.3.9) $N(B,\Gamma) = \sum_s 1_{B\times\Gamma}(s,\Delta Y_s) \qquad \Delta Y_s = Y_s - Y_{s-}$,

(3.3.10) $\nu(B,\Gamma) = E(N(B,\Gamma))$,

$$\int_0^t \int_{\mathbb{R}_n^o} |\underline{x}|^2 \wedge 1\, \nu\,(ds,d\underline{x}) < \infty \qquad t \in T$$

and $\nu_t(\Gamma) = \nu([0,t],\Gamma)$ is increasing and continuous.

From (3.3.5) we have that for $i,j=1,\ldots,n$ and $t \in T$

(3.3.11) $$\mu_{ij}(t) = EX_i(t)X_j(t) = \sigma_{ij}(t) + \lambda_{ij}(t) < \infty$$

where

$$\lambda_{ij}(t) = \int_{\mathbb{R}_n^o} x_i x_j \nu(t,d\underline{x}) \qquad \underline{x} = (x_1,\ldots,x_n) .$$

Define

(3.3.12) $$\mu_o(t) = \sum_{i=1}^n \sigma_{ii}(t) + \sum_{i=1}^n \lambda_{ii}(t)$$

and denote by μ_{ij}, σ_{ij}, λ_{ij} and μ_o the corresponding finite (signed) measures on (T,A) generated by $\mu_{ij}(t)$, $\sigma_{ij}(t)$, $\lambda_{ij}(t)$ and $\mu_o(t)$ respectively.

Then for each $i,j=1,\dots,n$ $\lambda_{ij} \ll \mu_o$, $\sigma_{ij} \ll \mu_o$, $\mu_{ij} \ll \mu_o$ and

(3.3.13) $$R_1(t) = \left\{\frac{d\sigma_{ij}}{d\mu_o}(t)\right\}_{ij,=1}^{n} \quad \text{a.e.} \quad d\mu_o(t)$$

(3.3.14) $$R_2(t) = \left\{\frac{d\lambda_{ij}}{d\mu_o}(t)\right\}_{i,j=1}^{n} \quad \text{a.e.} \quad d\mu_o(t)$$

are non-negative definite matrices a.e. $d\mu_o(t)$ and so is the matrix $R(t) = (r_{ij}(t))_{i,j=1}^{n}$, where for $i,j=1,\dots,n$

(3.3.15) $$r_{ij}(t) = \frac{d\mu_{ij}}{d\mu_o}(t) = \frac{d\sigma_{ij}}{d\mu_o}(t) + \frac{d\lambda_{ij}}{d\mu_o}(t) \quad \text{a.e.} \quad d\mu_o(t)\,.$$

Finally if $A(t)$ is an $n\times n$ non-negative definite matrix a.e. $d\mu_o(t)$, we denote by $L_A^2(\mu_o)$ the linear space of functions

(3.3.16) $$L_A^2 = \{f:T\to \mathbb{R}^n:\ \int_T f(t)'A(t)f(t)d\mu_o(t) < \infty\}.$$

In the next two results we use Theorem 3.3.1 to identify some functionals of the process Y_t as elements of an appropriate Hilbert space H_Y and its exponential space $EXP(H_Y)$. Having this and using the framework of Chapter II, we will be able to study symmetric tensor product measures and multiple stochastic integrals of the independently scattered measures X_i's where $X_i([0,t]) = X_i(t)$ $i=1,\dots,n$ and the latter are given in (3.3.4).

Proposition 3.3.1 Let Y_t, $t\in T$ be a stochastic process as in Assumption 3.3.1. Let H_W be the Gaussian space generated by W_t, i.e.

$$H_W = \overline{sp}\{\underline{a}' W_t;\ \underline{a}\in \mathbb{R}_n,\ t\in T\}$$

and H_q the "Poisson" space generated by q, i.e.

(3.3.17) $$H_q = \{I_q(g):\ g\in L^2(T\times \mathbb{R}_n^o,\ A\times B_n^o,\nu)\}.$$

Define

$$H_Y = H_W \oplus H_q \ .$$

Let $R(t)$ be as in (3.3.15). Then if $f \in L^2_R(\mu_o)$, $f(t) = (f_1(t),\ldots,f_n(t))$, the random variable

$$\int_T f\cdot dY = \sum_{i=1}^{n} I_{W_i}(f_i) + I_q(f'\underline{x}) \tag{3.3.18}$$

is an element of H_Y, where $(f'\underline{x})(t,\underline{x}) = f(t)'\underline{x}$ $t \in T$, $\underline{x} \in \mathbb{R}_n$, and $I_X(\phi)$ is the isometric integral (Theorem 2.1.1) of ϕ w.r.t. the orthogonally scattered mesure X. Moreover,

$$f(t)'\underline{x} \in L^2(T\times \mathbb{R}^o_n, A\times B^o_n, \nu) \cap L^1(T\times \mathbb{R}^o_n, A\times B^o_n, \nu)\ .$$

<u>Proof</u> To prove that $\int_T f\cdot dY \in H_Y$ it is enough to show that $f_i \in L^2(T,A,\sigma_{ii})$ $i=1,\ldots,n$ and $f'\underline{x} \in L^2(T\times \mathbb{R}^o_n, A\times B^o_n,\nu)$, since $I_{W_i}(f_i) \in H_W$ $i=1,\ldots,n$ and $I_q(f'\underline{x}) \in H_q$.

Since $f \in L^2_R(\mu_o)$ then from (3.3.15) $f \in L^2_{R_1}(\mu_o) \cap L^2_{R_2}(\mu_o)$ where R_1 and R_2 are given in (3.3.13) and (3.3.14) respectively. Then

$$\sum_{i=1}^{n} \int_T f_i^2(t)\, d\sigma_{ii}(t) = \sum_{i=1}^{n} \int_T f_i(t) \frac{d\sigma_{ii}}{d\mu_o}(t) f_i(t)\, d\mu_o < \infty$$

i.e. $\quad f_i \in L^2(T,A,\sigma_{ii}) \quad i=1,\ldots,n.$

On the other hand, using (3.3.12)

$$\int_T \int_{\mathbb{R}^o_n} (f(t)'\underline{x})^2 d\nu(t,\underline{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \int_T \int_{\mathbb{R}^o_n} f_i(t) f_j(t) x_i x_j\, d\nu(t,\underline{x}) \tag{3.3.19}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_T f_i(t) f_j(t)\, d\lambda_{ij}(t) = \int_T f(t)' R_2(t) f(t)\, d\mu_o(t) < \infty\ .$$

Finally, to prove that $f(t)'\underline{x} \in L^1(T\times \mathbb{R}^o_n, A\times B^o_n,\nu)$ it is enough to show that

$f_i(t)x_i \in L^1(T\times \mathbb{R}_n^o, A\times B_n^o, \nu)$ i=1,...,n since $f(t)'\underline{x} = \sum_{i=1}^n f_i(t)x_i$. By (3.3.19) for each i=1,...,n

$$\int_T f_i^2(t)d\lambda_i(t) < \infty.$$

Then since each λ_i is a finite measure on (T,A)

$$\int_T |f_i(t)|d\lambda_i(t) < \infty$$

and therefore using (3.3.12)

$$\int_T \int_{\mathbb{R}_n^o} f_i(t)x_i d\nu(t,\underline{x}) = \int_T f_i(t)d\lambda_i(t) < \infty .$$

Q.E.D.

For the process Y_t, Theorem 3.3.1 is written in the following manner.

Proposition 3.3.2 Under the assumptions of Proposition 3.3.1

(3.3.20) $$\mathrm{EXP}(H_Y) \overset{\gamma}{\cong} L^2(\Omega, F^Y, P)$$

where γ is given by (3.3.2) in Theorem 3.3.1 and

$$F^Y = \sigma(Y_t : t\in T) \vee \{\text{P-null sets of } \Omega\}.$$

Proof From the construction of the Levy-Itô decomposition (3.3.6) of Y_t (Gikhman and Skorokhod (1969)) W_t and q are F^Y measurables. On the other hand, by (3.3.6) and Proposition 3.3.1, Y_t is $F^W \vee F^q$-measurable all $t\in T$, where $F^W = \sigma(H_W)$ and $F^q = \sigma(H_q)$. Then the result follows using Theorem 3.3.1 and the fact that H_W and H_q are independent.

Q.E.D.

For purposes of later reference, the main properties of the integral $\int_T f\cdot dY$ defined in Proposition 3.3.1 are summarized in the next result.

Lemma 3.3.1 Under the assumptions of Proposition 3.3.1, for $f\in L_R^2(\mu_o)$ define

(3.3.21) $$F_t = \int_{[0,t]} f\cdot dY = \int_T 1_{[0,t]}(s)f(s)\cdot dY_s \quad t\in T$$

where right hand side is defined in (3.3.18). Then $(F_t)_{t\in T}$ is a zero mean L^2-stochastic process with independent increments and (F_t, F_t^Y) is an L^2-right continuous martingale such that

a) $$F_t \in H_Y \qquad t\in T\ .$$

b) $$\langle F\rangle_t = \int_0^t f(s)'R(s)f(s)d\mu_o(s) < \infty \quad \forall\ t\in T$$

where $\langle F\rangle$ denotes the predictable quadratic variation of F.

c) If $g \in L_R^2(\mu_o)$

$$\langle F,G\rangle_t = \int_0^t f(s)'R(s)g(s)d\mu_o(s) = E(F_tG_t) \quad t\in T\ .$$

d) The characteristic function of F_t is given by

$$\Phi_{F_t}(a) = \exp\{-\tfrac{1}{2}a\ \langle F^c\rangle_t + \int_0^t \int_{\mathbb{R}_o^n}(e^{iaf'(s)\underline{x}}-1-iaf'(s)\underline{x})d\nu(s,\underline{x})$$

where

(3.3.22) $$\langle F^c\rangle_t = \int_0^t f(s)'R_1(s)f(s)d\mu_o(s) \qquad t\in T.$$

<u>Proof</u> Since the process Y_t is right continuous, then (F_t^Y) $t\in T$ is a right continuous filtration. From Proposition 3.3.2 $F_t^Y \subset F^Y = \sigma(H_W)\vee\sigma(H_q)$ all $t\in T$ and by (3.3.18)

$$F_t = \sum_{i=1}^n I_{W_i}(1_{[0,t]}f_i) + \int_T \int_{\mathbb{R}_o^n} f'(s)\underline{x}\ 1_{[0,t]}(s)dq(s,\underline{x}) \in H_Y\ .$$

It is known (Galthouck (1976)) that if $f_i \in L^2(T,A,\sigma_{ii})$ $i=1,\dots,n$ and

(3.3.23) $$F_t^c = \sum_{i=1}^n I_{W_i}(1_{[0,t]}f_i) \quad f(t) = (f_1(t),\dots,f_n(t))$$

then $(F_t^c, F_t^Y)_{t\in T}$ is an L^2-continuous martingale with $\langle F^c\rangle_t$ given by (3.3.22). Also it is known that if

$$F_t^d = \int_T \int_{\mathbb{R}_n^o} f(s)'\underline{x}\, 1_{[0,t]}(s)dq(s,\underline{x}) \tag{3.3.24}$$

then (F_t^d, F_t^Y) is an L^2-right continuous martingale with

$$\langle F^d\rangle_t = \int_0^t \int_{\mathbb{R}_n^o} (f(s)'\underline{x})^2 d\nu(s,\underline{x}) \qquad t \in T.$$

But from (3.3.19) and the last expression

$$\langle F^d\rangle_t = \int_0^t f(s)'R_2(s)f(s)d\mu_o(s) \qquad t \in T. \tag{3.3.25}$$

Thus (F_t, F_t^Y) is an L^2-right continuous martingale and since $F_t^c \in H_W$ and $F_t^d \in H_q$, then $E(F_t) = 0 \quad t \in T$ and

$$\langle F\rangle_t = E(F_t^2) = E(F_t^c)^2 + E(F_t^d)^2 = \langle F^c\rangle_t + \langle F^d\rangle_t$$

$$= \int_0^t f(s)'R(s)f(s)d\mu_o(s) < \infty \qquad t \in T .$$

The proof of (c) follows from (b) and the polarization identity

$$\langle F,G\rangle_t = \frac{1}{4}\{\langle F+G,F+G\rangle_t - \langle F-G,F-G\rangle_t\} \qquad t \in T .$$

Finally (d) follows since $F_t = F_t^c + F_t^d$, $F_t^c \in H_W$, $F_t^d \in H_q$, H_W and H_q are independent and using (3.3.5).

Q.E.D.

Now we shall apply Propositions 3.3.1 and 3.3.2 and our results in Chapter 2 to construct an $L^2(\Omega)$-valued product stochastic measure of $X_1, \ldots, X_n$.

The next theorem gives an $L^2(\Omega)$-valued product stochastic measure of non-identically distributed independently scattered measures. It is the main result of this section.

Theorem 3.3.2 Let $\{X_1,\ldots,X_n\}$ be a system of $n \geq 1$ L^2-independently

scattered measures on $(T,\mathcal{A})$ such that $Y_t = (X_1(t),\dots,X_n(t))$ $t \in T$, is an n-dimensional stochastic process with independent increments as in Assumption 3.3.1, where $X_i(t) = X_i([0,t])$. Then there exists a unique $L^2(\Omega,F^Y,P)$-valued measure $\overset{n}{\underset{i=1}{\otimes}} X_i$ on $(T^n,\mathcal{A}^n)$ such that for $A_i \in \mathcal{A}$ $i=1,\dots,n$

$$\overset{n}{\underset{i=1}{\otimes}} X_i(A_1\times\dots\times A_n) = X_1(A_1)\otimes\dots\otimes X_n(A_n) \tag{3.3.26}$$

and for $A \in \mathcal{A}^n$ $\qquad \overset{n}{\underset{i=1}{\otimes}} X_i(A) \in H_Y^{\otimes n}$

$$E(\overset{n}{\underset{i=1}{\otimes}} X_i(A)) = 0 \tag{3.3.27}$$

and

$$\mathrm{VAR}(\overset{n}{\underset{i=1}{\otimes}} X_i(A)) = \frac{1}{n!}\sum_{\Pi} \mu_{1\Pi_1}\otimes\dots\otimes\mu_{n\Pi_n}(A\cap A^{\Pi}) \tag{3.3.28}$$

where

$$\mu_{ij}(C\cap B) = EX_i(C)X_j(B) \qquad C,B \in \mathcal{A} \quad i,j=1,\dots,n. \tag{3.3.29}$$

<u>*Proof*</u> *Let* $\{\underline{e}_i\}_{i=1}^n$ *be the canonical basis in* $\mathbb{R}_n$ *and define*

$$f_A^i(t) = 1_A(t)\underline{e}_i \qquad t \in T \qquad A \in \mathcal{A} \qquad i=1,\dots,n .$$

Then for each $i=1,\dots,n$ $f_A^i \in L_R^2(\mu_o)$ since

$$\int_T f_A^i(t)'R(t)f_A^i(t)d\mu_o(t) = \int_A r_{ii}(t)d\mu_o(t) = \int_A d\mu_i(t)$$

$$= \mu_i(A) < \infty.$$

Next, by Proposition 3.3.1 for each $A \in \mathcal{A}$

$$X_i(A) = \int_T f_A^i \cdot dY \in H_Y \qquad\qquad i=1,\dots,n$$

where $H_Y = H_W \oplus H_q$. Thus each independently scattered measure X_i is an orthogonally scattered measure on $(T,\mathcal{A})$ with values in the common Hilbert spcae H_Y and control measure μ_i. Therefore existence and uniqueness of the $H_Y^{\otimes n}$-valued measure $\overset{n}{\underset{i=1}{\otimes}} X_i$ follow by Theorem 2.2.1. On the other hand

by Proposition 3.3.2 we can see $\bigodot_{i=1}^{n} X_i$ as an $L^2(\Omega,F^Y,P)$-valued measure. Finally, the equalities (3.3.27) and (3.3.28) follow in the same way as for $\bigodot_{i=1}^{n} W_i$ in Proposition 3.1.1.

Q.E.D

In order to compute the symmetric tensor product measure $\bigodot_{i=1}^{n} X_i(A)$ for $A \in A^n$, we have to find concrete expressions for general symmetric tensor products of elements in H_Y where Y is an n-dimensional stochastic process with independent increments as in (3.3.4). This last problem was studied by Kailath and Segall (1976) for the case of a one-dimensional stochastic process with stationary and independent increments. Although the main ideas behind the next two results originated from the above named work, we were not able to find them in the literature in the generality that they are presented and proven here.

The next result is a generalization of Propositions 3.1.2 and 3.2.2. We remark that for this theorem to hold it is required that T is an interval of the real line.

<u>Theorem 3.3.3</u> Let (Y_t) $t \in T = [0,T_o]$ and H_Y be as in Proposition 3.3.1. For each i=1,...,n let $f^i \in L^2_R(\mu_o)$ and

$$F_i(t) = \int_{[0,t]} f^i \cdot dY, \quad F_i = \int_T f^i \cdot dY \tag{3.3.30}$$

where the f^i's are not necessarily all different. Then

$$F_1 \bigodot \ldots \bigodot F_n = (n!)^{\frac{1}{2}} \underline{P}^n_{T_o}(F_1,\ldots F_n) \tag{3.3.31}$$

where $\underline{P}^n(F_1,\ldots,F_n)$ are multivariate functionals of the process Y (Kailath and Segall (1976), Meyer (1976)) defined by

$$\underline{P}^o_t(\cdot) = 1 \qquad t \in T \tag{3.3.32}$$

(3.3.33) $\underline{P}_t^1(F_i) = F_i(t) \quad t \in T \qquad i=1,\ldots,n$

(3.3.34) $\underline{P}_t^n(F_1,\ldots,F_n) =$

$$\frac{1}{n} \sum_{r=1}^{n} \int_{[0,t]} \underline{P}_{s-}^{n-1}(F_1,\ldots,F_{r-1},F_{r+1},\ldots,F_n)dF_r(s) \quad t \in T.$$

The proof of this theorem is based on the following lemma.

Lemma 3.3.2 Let Y_t $t \in T$ and H_Y be as in Theorem 3.3.3, $f \in L_R^2(\mu_o)$, $F(t) = \int_{[0,t]} f \cdot dY$ and $F = F_{T_o} = \int_T f \cdot dY$. Then

$$F^{\circledcirc n} = (n!)^{\frac{1}{2}} P_{T_o}^n (F)$$

where $P^n(F)$ $n \geq 1$ are univariate functionals of the process Y (Kailath and Segall (1976), Meyer (1976)) defined by

(3.3.35) $P_t^o(\cdot) = 1 \qquad \forall\ t \in T$

(3.3.36) $P_t^1(F) = F(t)$

(3.3.37) $P_t^n(F) = \int_{[0,t]} P_{s-}^{n-1}(F)dF(s).$

Proof We first show that the functionals (3.3.35)-(3.3.37) belong to $L^2(\Omega,F^Y,P)$ and that they are square integrable F_t^Y-martingales: each $P_t^n(F)$ is F_t^Y adapted all $n \geq 1$ $t \in T$ and by Lemma 3.3.1 $(P_t^1(F),F_t^Y)$ is an L^2-martingale. Next using Lemma 3.3.1 (b) we have that

$$E\left(\int_{[0,t]} [P_{s-}^{n-1}(F)]^2 \, d\langle F\rangle_s\right)$$

$$= \int_{[0,t]} E(P_s^{n-1}(F))^2 f(s)'R(s)f(s)d\mu_o(s)$$

$$\leq E(P_{T_o}^{n-1}(F))^2 \int_{[0,t]} f(s)'R(s)f(s)d\mu_o(s).$$

Hence, using induction, if $E(P_{T_o}^{n-1}(F))^2 < \infty$, then

$$E\left(\int_T (P^{n-1}_{s-}(F))^2 d\langle F\rangle_s\right) < \infty$$

which shows that $P^n_t(F) = \int_{[0,t]} P^{n-1}_{s-}(F)\,dF(s)$ is a square integrable F^Y_t-martingale.

Next from Lemma 3.3.1 amd Theorem 3.3.1, if $f \in L^2_R(\mu_o)$

and $F_{T_o} = F^c_{T_o} + F^d_{T_o} \in H_W \oplus H_q$

$$\gamma(\exp \circledcirc(F)) = \exp\left(F^c_{T_o} - \frac{1}{2}E\,(F^c_{T_o})^2 \prod_{s\le T_o} (1+f'(s)\Delta Y_s)e^{-\int_T\int_{\mathbb{R}^o_n} f'(u)\underline{x}d\nu(u,\underline{x})}\right) \tag{3.3.38}$$

since $f'(s)\underline{x} \in L^2(T\times \mathbb{R}^o_n, A\times B^o_n, \nu) \cap L^1(T\times \mathbb{R}^o_n, A\times B^o_n, \nu)$ by Proposition 3.3.1. But under this last condition

$$F^d_{T_o} = I_q(f'(s)\underline{x}) = \int_T \int_{\mathbb{R}^o_n} f'(s)\underline{x}dN(s,\underline{x}) - \int_T \int_{\mathbb{R}^o_n} f'(s)\underline{x}d\nu(s,\underline{x}).$$

Then using (3.3.9)

$$\begin{aligned} F^d_{T_o} &= \sum_{s\le T_o} f'(s)\Delta Y_s - \int_T \int_{\mathbb{R}^o_n} f'(s)\underline{x}d\nu(s,\underline{x}) \\ &= \sum_{s\le T} \Delta F_s - \int_T \int_{\mathbb{R}^o_n} f'(s)\underline{x}d\nu(s,\underline{x}) \end{aligned}$$

and therefore from the last expression and (3.3.38)

$$\gamma(\exp \circledcirc(F)) = \exp\left(F^c_{T_o} + F^d_{T_o} - \frac{1}{2}\langle F^c\rangle_{T_o}\right) \prod_{s\le T_o} \{(1+\Delta F_s)e^{-\Delta F_s}\} = \mathcal{E}xp(F)$$

where $\mathcal{E}xp(F)$ is the exponential semimartingale of F (Doleans-Dade (1970)). Then the lemma follows using Proposition 3.3.2, since from Section 3 in Doleans-Dade (1970), for any $\beta \in \mathbb{C}$

$$\mathcal{E}xp(\beta F)_t = \sum_{n=0}^{\infty} \beta^n P^n_t(F) \quad \text{a.s.} \quad \forall\, t \in T \tag{3.3.39}$$

and on the other hand

$$\exp \circledcirc(\beta F) = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \beta^n F^{\circledcirc n}.$$

Q.E.D.

<u>Proof of Theorem 3.3.3</u> Using the notation in (2.2.7) and by Lemma 3.3.2

$$F_1 \circledcirc \cdots \circledcirc F_n = (n!)^{-\frac{1}{2}} \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{M \in \mathcal{P}_\ell} P^n_{T_o}\left(\sum_{i=1}^{n} 1_{M^c}(i) F_i\right).$$

Next, using induction on n one shows that for all t

$$\underline{P}^n_t(F_1,\ldots,F_n) = \frac{1}{n!} \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{M \in \mathcal{P}_\ell} P^n_t\left(\sum_{i=1}^{n} 1_{M^c}(i) F_i\right)$$

from which the theorem follows.

Q.E.D.

We now obtain an extension of Proposition 3.1.2.

<u>Lemma 3.3.3</u> Let Y_t $t \in T$ and H_Y be as in Theorem 3.3.3. Let $f^1,\ldots,f^k$ be elements in $L_R(\mu_o)$ with disjoint support,

$$F_i(t) = \int_{[0,t]} f^i \cdot dY \text{ and } F_i = F_i(T_o) = \int_T f^i \cdot dY.$$

Then

$$F_1^{\circledcirc n_1} \circledcirc \cdots \circledcirc F_k^{\circledcirc n_k} = (n!)^{-\frac{1}{2}} \prod_{i=1}^{k} P^{n_i}_{T_o}(F_i)\,(n_i!)^{\frac{1}{2}} \tag{3.3.40}$$

where P^{n_i} are defined in (3.3.35)-(3.3.37) and $n = \sum_{i=1}^{k} n_i$.

<u>Proof</u> Since $f^1,\ldots,f^k$ have disjoint support, then by Lemma 3.3.1 (c), $F_1,\ldots,F_k$ are mutually orthogonal elements in H_Y and therefore the family

$$\{F_1^{\circledcirc n_1} \circledcirc \cdots \circledcirc F_k^{\circledcirc n_k} : n_1 \geq 0,\ldots,n_k \geq 0\}$$

is orthogonal in $\mathrm{EXP}(H_Y) \overset{Y}{\cong} L^2(\Omega, F^Y, P)$.

Next, for $\beta_1,\ldots,\beta_k \in \mathbb{R}$

$$\exp \circledcirc\left(\sum_{i=1}^{k} \beta_i F_i\right) = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \left(\sum_{i=1}^{k} \beta_i F_i\right)^{\circledcirc n} \tag{3.3.41}$$

$$= \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \sum_{n_1+\cdots+n_k=n} \frac{\beta_1^{n_1}}{n_1!} \cdots \frac{\beta_k^{n_k}}{n_k!} F_1^{\circledcirc n_1} \circledcirc \cdots \circledcirc F_k^{\circledcirc n_k}$$

$$= \sum_{n_1 \ldots n_k} \frac{((n_1+\ldots+n_k)!)^{-\frac{1}{2}}}{n_1! \ldots n_k!} \beta_1^{n_1} \ldots \beta_k^{n_k} F_1^{\circledcirc n_1} \circledcirc \ldots \circledcirc F_k^{\circledcirc n_k} .$$

On the other hand since for $i \neq j$ f^i and f^j have disjoint support, then if $[\ ,\ ]_t$ denotes the optional quadratic variation, using Lemma 3.3.1 (b) we obtain

$$\begin{aligned}[][\beta_i F_i, \beta_j F_j]_t &= \beta_i \beta_j \langle F_i^c, F_j^c \rangle_t + \beta_i \beta_j \sum_{s \le t} \Delta F_i(s) \Delta F_j(s) \\ &= \beta_i \beta_j \sum_{s \le t} f^i(s)' \Delta Y(s) f^j(s) \Delta Y(s) = 0 .\end{aligned}$$

Then

$$\mathcal{E}xp(\beta_1 F_1 + \ldots + \beta_k F_k)_t = \prod_{i=1}^{k} \mathcal{E}xp(\beta_i F_i)_t \quad \text{a.s.} \quad \forall\, t \in T. \tag{3.3.42}$$

Hence the assertion of the lemma follows from (3.3.39), since

$$F_1^{\circledcirc n_1} \circledcirc \ldots \circledcirc F_k^{\circledcirc n_k} = (n!)^{-\frac{1}{2}} \left(\frac{\partial^{n_1+\ldots+n_k}}{\partial \beta_1^{n_1} \ldots \partial \beta_k^{n_k}} \right)_{\underline{\beta}=0} \prod_{i=1}^{k} \mathcal{E}xp(\beta_i F_i)_{T_o}$$

$$\underline{\beta} = (\beta_1, \ldots, \beta_k).$$

Q.E.D

With the above lemma we are able to compute the symmetric tensor product measure $\circledcirc_{i=1}^{n} X_i(A)$, for special sets $A \in \mathcal{A}^n$. The following result will be used in Section 3.4.

<u>Corollary 3.3.1</u> Let $\{X_1, \ldots, X_n\}$ be a system of independently scattered measures as in Theorem 3.3.2. Let $A_1, \ldots, A_k$ be disjoint sets in $\mathcal{A}$ for $k > 0$. Then if $i_1, \ldots, i_k \in \{1, \ldots, n\}$

$$X_{i_1}(A_1) \circledcirc \ldots \circledcirc X_{i_k}(A_k) = (k!)^{-\frac{1}{2}} X_{i_1}(A_1) \ldots X_{i_k}(A_k). \tag{3.3.43}$$

The proof follows by Lemma 3.3.3 since $P^1_{T_o}(F_i) = F_i(T_o)$ and by taking

$f^i(t) = 1_{A_i}(t)\underline{e}_i$, where $\{\underline{e}_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$.

The main properties of the functionals $\underline{P}^n(Z_1,\ldots,Z_n)$ and $P^n(Z)$ defined in (3.3.32)-(3.3.34) and (3.3.35)-(3.3.37) respectively, are given in Kailath and Segall (1976) and Meyer (1976), including recursive expressions to compute them. In particular, the univariate functionals $P^n(Z)$ are Hermite polynomials in the case when Z is a Gaussian martingale.

The first few expressions for $\underline{P}^n(Z_1,\ldots,Z_n)$ are:

$$\underline{P}_t^0(\cdot) = 1 \qquad \underline{P}_t^1(Z_i) = Z_i(t) \quad \forall \in T$$

(3.3.44) $$\underline{P}_t^2(Z_i,Z_j) = \tfrac{1}{2}\{Z_i(t)Z_j(t) - [Z_i,Z_j]_t\}$$

(3.3.45) $$\underline{P}_t^3(Z_i,Z_j,Z_k) = \tfrac{1}{6}\{Z_i(t)Z_j(t)Z_k(t)$$

$$- Z_i(t)[Z_j,Z_k]_t - Z_j(t)[Z_i,Z_k]_t$$

$$- Z_k(t)[Z_i,Z_j]_t + 2\sum_{s\le t}\Delta Z_i(s)\Delta Z_j(s)\Delta Z_k(s)\}$$

where $Z_1,\ldots,Z_n$ are semimartingales not necessarily all distinct.

<u>Multiple stochastic integrals</u> As in the Gaussian and Poisson cases (Sections 3.1 and 3.2 respectively), integrals with respect to $\bigotimes_{i=1}^n X_i$ can be constructed using the theory of Section 2.3. Under Assumption 3.1.1 T is an interval of the real line and the measures μ_i's are non-atomic. Then by Theorem 2.3.2 a function f is $\bigotimes_{i=1}^n X_i$-integrable if and only if $f \in L^2(T^n, A^n, \bigotimes_{i=1}^n \mu_i)$, i.e. $L_1(\bigotimes_{i=1}^n X_i) = L^2(T^n, A^n, \bigotimes_{i=1}^n \mu_i)$.

Thus from Proposition 3.3.2 we have that if $f \in L^2(T^n, A^n, \bigotimes_{i=1}^n \mu_i)$

$$I_n(f;X_1,\ldots,X_n) = \int_{T^n} f(\underline{t})\, d\bigotimes_{i=1}^n X_i(\underline{t})$$

is an element of $L^2(\Omega,F^Y,P)$ with all the properties of the integral of Section 2.3. In the next result we summarize some of these properties. We use the notation of Theorem 2.3.3.

Proposition 3.3.3 Let $X_1,\ldots,X_n$ and $\overset{n}{\underset{i=1}{\otimes}} X_i$ be as in Theorem 3.3.2. Then a function f is $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrable if and only if $f \in L^2(T^n,A^n,\overset{n}{\underset{i=1}{\otimes}} \mu_i)$ in which case

a) $$I_n(f;X_1,\ldots,X_n) \in L^2(\Omega,F^Y,P).$$

b) $$E(I_n(f;X_1,\ldots,X_n)) = 0.$$

c) If $g \in L^2(T^m,A^m,\overset{m}{\underset{i=1}{\otimes}} \mu_i)$ $m \neq n$

$$E(I_n(f;X_1,\ldots,X_n)I_m(g;X_1,\ldots,X_m))$$

$$= \delta_{nm} \int_{T^n} f_{\otimes n}(\underline{t})' R^{\otimes n}(\underline{t})\ g_{\otimes n}(\underline{t}) d\mu_o^n(\underline{t}).$$

The proof follows analogous to the proof of Lemma 3.1.1.

One dimensional case We now consider the case when $X = X_1 = \ldots = X_n$ i.e. $Y_t = X_t$ is a one dimensional stochastic process with independent increments as in Assumption 3.1.1 with n=1. Then X is an H_X-valued orthogonally scattered measure and by (3.3.11) it has control measure

$$\mu(A) = \sigma_1(A) + \int_A \int_{\mathbb{R}^o} x^2 d\nu(t,x) \qquad A \in A \ . \tag{3.3.46}$$

Then in this case $\mu_o = \mu$ and $r(t) = 1$ all $t \in T$. If we identify functions which are equal a.e. $d\mu_o(t)$, the space of functions $\mathcal{L}_R^2(\mu_o)$ can be taken to be the Hilbert space $L_R^2(\mu_o)$, that in this situation is equal to $L^2(T,A,\mu)$. Then is this case

$$I_X(f) = \int_T f \cdot dY$$

where the last integral is defined in (3.3.18) for n=1.

Following the notation of Propositions 2.2.1 and 2.3.6, by Proposition 3.3.2 we obtain that for $n\geq 1$ the symmetric tensor product stochastic measure $X^{\circledast n}$ on (T^n, A^n) and the multiple stochastic integral $I_{n\circledast}(f;X)$, $f \in L^2(T^n, A^n, \mu^{\circledast n})$, are $L^2(\Omega, F^X, P)$-valued elements. They satisfy the properties of Proposition 2.2.1 and Proposition 2.3.5 respectively.

The next two results are extensions of Theorem 3.1 in Itô (1951) (Propositions 3.1.5 and 3.1.6 in our work) to the case of a general L^2-independent increments process.

<u>Proposition 3.3.4</u> Let X be a one dimensional stochastic process with independent increments as above. Let $P^k(\cdot)$ be the functionals defined in (3.3.35)-(3.3.37). Assume that $f_1(t),\ldots,f_m(t)$ is a system of real valued functions in $L^2(T,A,\mu)$ with disjoint support. Define

$$f(\underline{t}) = f_1(t_1)\ldots f_1(t_{k_1})f_2(t_{k_1+1})\ldots f_2(t_{k_1+k_2})\ldots f_m(t_{k_1+\ldots+k_m-1})\ldots f_m(t_{k_1+\ldots k_m}) .$$

Then if $n = k_1+\ldots+k_m$

$$I_{n\circledast}(f;X) = (n!)^{-\frac{1}{2}} \prod_{i=1}^{m} P_{T_o}^{k_i}(F_i)(k_i!)$$

where $F_i = \int_T f_i dX = I_X(f_i)$.

<u>Proof</u> By Lemma 2.3.1 f is $X^{\circledast}$-integrable and

$$I_{n\circledast}(f;X) = I_X(f_1)^{\circledast k_1}\circledast\ldots\circledast I_X(f_m)^{\circledast k_m} .$$

Next using Lemma 3.3.1 (c)

$$E(F_iF_j) = \int_T f_i(s)f_j(s)d\mu(s) = 0$$

since f_i, f_j have disjoint support. Then by Lemma 3.3.3

$$I_X(f_1)^{\otimes k_1} \otimes \ldots \otimes I_X(f_m)^{\otimes k_m} = (n!)^{-\frac{1}{2}} \prod_{j=1}^{m} P_{T_o}^{k_i}(F_i)(k_i!) \ .$$

Q.E.D.

Corollary 3.3.2 Let $T = [0,1]$. Then if

$$T_1^n = \{(t_1,\ldots,t_n) : 0 \le t_1 < \ldots < t_n \le 1\}$$

$$I_{n\otimes}(1_{T_1^n};X) = (n!)^{-\frac{1}{2}} P_1^n(X)$$

That is, formally

$$\int\cdots\int_{0\le t_1<\ldots<t_n\le 1} dX(t_1)\ldots dX(t_n) = (n!)^{-\frac{1}{2}} P_1^n(X) .$$

Proof By Definition 2.3.1, $I_{n\otimes}(1_{T_1^n};X) = X^{\otimes n}(T_1^n)$. Then the result follows from the last proposition since $X^{\otimes n}$ is finitely additive, μ is continuous and Proposition 2.2.1 (c).

Q.E.D.

Finally, in this section we discuss the completeness of the multiple stochastic integrals $I_n\otimes(f;X)$ $n\ge 0$ in $L^2(\Omega,F^X,P)$, obtaining a characterization of the Gaussian and certain Poisson random measures on (T,A).

Proposition 3.3.5 Let X be an L^2-stochastic process with independent increments as in Proposition 3.3.4. Then the system

(3.3.47) $$\{I_{n\otimes}(f_n:X) : f_n \in L^2(T^n,A^n,\mu^{\otimes n}), \quad n\ge 1\}$$

generates $L^2(\Omega,F^X,P)$ if and only if

a) $\nu(t,A) = 0 \quad \forall\, A \in B^o,\ t \in T$ (Gaussian situation) or

b) $\sigma_1(t) = 0$ and $\nu(t,\cdot)$ is concentrated in one point $x_o \neq 0$.

Proof Assume (a) holds. Then $\mu(A) = \sigma_1(A)$ $A \in A$, $X_t = W_t$ $t \in T$, $I_X(f) = I_W(f)$ $f \in L^2(T,A,\mu)$ and $H_X = H_W$. Thus from Proposition 2.3.7 the system (3.3.47) generates the space $EXP(H_W)$ and by (3.1.4) $EXP(H_W)$ is identified with $L^2(\Omega,F^W,P)$. Then the system (3.3.47) generates $L^2(\Omega,F^W P)$.

Now assume (b) holds, then $\mu(A) = x_o^2\nu(A;x_o)$ $A \in A$ for $x_o \neq 0$, $q(A) = q(A;x_o)$ is a Poisson random measure on (T,A), $X_t = q([0,t])$, $I_X = I_q(f)$ $f \in L^2(T,A,\mu)$ and $H_X = H_q$. Then as above, from Proposition 2.3.7 the system (3.3.47) generates the space $EXP(H_q)$ and by Lemma 3.2.2, $EXP(H_q)$ is identified with $L^2(\Omega,F^q,P)$. Then the system (3.3.47) generates $L^2(\Omega,F^q,P)$.

Next assume that the system (3.3.47) generates the space $L^2(\Omega,F^X,P)$ which is identified with $EXP(H)$ by Proposition 3.3.2, where $H = H_W \oplus H_q$. Then by Proposition 2.3.7 the system (3.3.47) generates $EXP(H_X)$. Therefore, $EXP(H_X) = EXP(H)$ which implies $H_X = H$. Then for each $g_1 \in L^2(T,A,\sigma_1)$ and $g_2 \in L^2(T \times \mathbb{R}^o, A \times B^o, \nu)$ there exists $f \in L^2(T,A,\mu)$ such that

$$I_W(g_1) + I_q(g_2) = I_X(f).$$

But from (3.3.18) in Proposition 3.3.1

$$I_X(f) = I_W(f) + I_q(fx)$$

Therefore

$$I_W(g_1) - I_W(f) = 0 \quad \text{a.e.}$$

and

$$I_q(g_2) - I_q(fx) = 0 \quad \text{a.e.}$$

which implies

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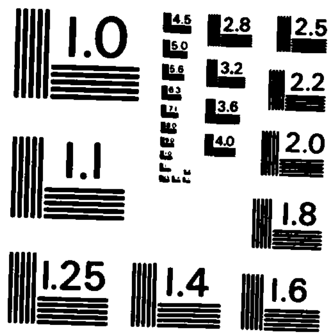
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0

$$(3.3.48) \qquad g_1 = f \quad \text{a.e. } d\sigma_1$$

and

$$(3.3.49) \qquad g_2(t,x) = f(t)x \quad \text{a.e. } d\nu\,.$$

Suppose there exists $t_1 \in T$ and $A \in \mathcal{B}^o$ such that

$$(3.3.50) \qquad 0 < \nu([0,t_1] \times A) < \infty\,.$$

Then $0 < \nu(t,A) < \infty \quad t > t_1$ since $\nu(t,A) = \nu([0,t] \times A)$ is a non-decreasing function. Let $g_1(t) = \alpha 1_{[0,t_1]}(t)$ for an arbitrary $\alpha \neq 0$ and $g_2(t,x) = 1_{[0,t_1]\times A}(t,x)$. Then $g_1 \in L^2(T,A,\sigma_1)$ and $g_2 \in L^2(T \times \mathbb{R}^o, A\times B^o, \nu)$ and by the above argument there exists $f \in L^2(T,A,\mu)$ such that

$$(3.3.51) \qquad f(t) = \alpha 1_{[0,t_1]}(t) \quad \text{a.e. } d\sigma_1$$

$$(3.3.52) \qquad f(t) = 1_{[0,t_1]\times A}(t,x) \quad \text{a.e. } d\nu\,.$$

Let $t \in [0,t_1]$ and $x_1, x_2 \in A$. Then $(t,x_1) \in [0,t_1] \times A$ and $(t,x_2) \in [0,t_1] \times A$ and by (3.3.52) $f(t)x_1 = f(t)x_2 = 1$ i.e. $f(t) \neq 0$ and $f(t) = 1/x_1 = 1/x_2$ which implies $x_1 = x_2$ and therefore $\nu(t,A)$ is concentrated in one point x_o, say, and $f(t) = 1/x_o \quad t \in [0,t_1]$. Then by (3.3.51) if $x_o \notin A^{-1}$, $\sigma_1(t) = 0$ $t \in [0,t_1]$ and since α is arbitrary, then $\sigma_1(t) = 0$ all $t \in [0,t_1]$. Then since $\nu(t,A)$ is non-decreasing the above argument holds for all $t > 0$, i.e. $\nu(t,A)$ is concentrated in one point and $\sigma_1(t) = 0$ all $t > 0$. Hence condition (b) holds.

On the other hand, if there do not exist $t_1 \in T$ and $A \in B^o$ such that (3.3.50) is satisfied, then $\nu(t,A) = 0$ all $t \in T$ and $A \in B^o$ since ν is a σ-finite measure on $(T \times \mathbb{R}^o, A \times B^o)$. Then condition (a) holds.

Q.E.D.

3.4 Comparisons with earlier results

Engel (1982) and Rosinski and Szulga (1982) have recently studied

L^2-valued product stochastic measures. In all these works the usual product of independently scattered measures has always been considered and in none of them has the symmetric tensor product been used. In this section we point out the main differences between the two approaches and the advantages of the symmetric tensor product measure.

Engel and Kakutani (Engel (1982)) have considered the construction of an $L^2(\Omega)$-valued product stochastic measure from a system of $n \geq 1$ stochastic processes $\{X_1(t),\ldots,X_n(t)\}$, $t \in T = [0,T_o]$, on a probability space (Ω,F,P), satisfying the following four conditions (R1-R4 in Engel's notation (1982)):

(3.4.1) The funtion $m_k(t) = EX_k(t)$ is a continuous function of bounded variation on T, for each $k=1,\ldots,n$.

(3.4.2) The function $\mu_k(t) = E(X_k(t)-m_k(t))^2$ is a continuous monotonely increasing function on T, for each $k=1,\ldots,n$.

(3.4.3) If $\{I_1,\ldots,I_q\}$ is any set of disjoint intervals contained in T and $\{i_1,\ldots,i_q\}$ is any set of integers where $1 \leq k \leq n$, $k=1,\ldots,q$, then $\{X_{i_1}(I_1),\ldots,X_{i_q}(I_q)\}$ forms an independent system of random variables, where for $I = (s,t]$, $X_i(I) = X_i(t) - X_i(s)$.

(3.4.4) If $I \subset T$ is any interval and $1 \leq j_1 < \ldots < j_k \leq n$ is any sequence of integers between 1 and n, then

$$E|X_{j_1}(I)X_{j_2}(I)\ldots X_{j_k}(I)|^2 < \infty$$

and

$$E(|X_{j_1}(I)\ldots X_{j_k}(I)|)^2 \to 0 \quad \text{as } |I| \to 0.$$

Let A_o^n (F_o^n in Engel's notation) be the field of elementary subsets of T^n of the form

$$B = \bigcup_{i_1 \ldots i_n = 1}^{q} c_{i_1 \ldots i_n} I_{i_1} \times \ldots \times I_{i_n} \tag{3.4.5}$$

where $\{I_1, \ldots, I_q\}$ is a partition of T into disjoint intervals (depending on B) for which $I_k < I_{k+1}$ (i.e. if $s_1 \in I_k$ and $s_2 \in I_{k+1}$ then $s_1 < s_2$) $k=1, \ldots, q-1$ and $c_{i_1 \ldots i_n}$ is either zero, indicating that $I_{i_1} \times \ldots \times I_{i_n}$ is not included in the union, or is one indicating that it is included. Engel (1982) defines the finitely additive $L^2(\Omega)$-valued product measure $Y^{(n)}$ on A_o^n as

$$Y^{(n)}(B) = \sum_{i_1 \ldots i_n = 1} c_{i_1 \ldots i_n} X_1(I_{i_1}) \ldots X_n(I_{i_n}) \qquad B \in A_o^n \tag{3.4.6}$$

and proves the following theorem which extends the measure Y^n to $A^n = \sigma(A_o^n)$.

<u>Theorem 3.4.1</u> (Theorem 4.5 Engel (1982)). Let $\{X_1(t), \ldots, X_n(t)\}$, $t \in T = [0, T_o]$ be a system of $n \geq 1$ stochastic processes satisfying the regularity conditions (3.4.1)-(3.4.4). Then the $L^2(\Omega)$-valued measure $Y^{(n)}$ defined by (3.4.6) on A_o^n can be extended to a countably $L^2(\Omega)$-valued measure (also denoted by $Y^{(n)}$) on the Borel σ-field $A^n = \sigma(A_o^n)$.

The idea of the proof of this theorem is to partition the set T^n into disjoint pieces on which an appropriate countably $L^2(\Omega)$-valued measure can be defined and then show that the sum of all these measures is the required measure. This procedure uses a complicated double induction and involves prior knowledge of what the measure $Y^{(n)}$ should look like, even when the mean function of each process is assumed to be zero. On the other hand the construction of the symmetric tensor product measure $\bigodot_{i=1}^{n} X_i$ follows the more natural ideas from the theory of product real valued measures. Moreover, the assumption that T is a subset of the real line is essential in the construction of Engel's product measure $Y^{(n)}$ and therefore more general

parameter sets T, as those considered in Sections 3.1 and 3.2 of this chapter, cannot be contemplated in Engel's framework.

Assumptions (3.4.2) and (3.4.3) imply that

$$Y_t = (X_1(t),\dots,X_n(t)) \quad t \in T$$

is an n-dimensional L^2-stochastic process with independent increments. Then if we assume $EX_i(t) = 0 \;\; \forall\, t \in T \;\; i=1,\dots,n$, we are able to construct the $L^2(\Omega,F^Y,P)$-valued product stochastic measure $\overset{n}{\underset{i=1}{\otimes}} X_i$ as in Section 3.3. This "zero mean" condition is satisfied in many interesting cases and allows us to use Hilbert space techniques in the construction of $\overset{n}{\underset{i=1}{\otimes}} X_i$. On the other hand, although the "zero mean" condition simplifies the proof of some of Engel's results (e.g. Theorem 4.1 Engel (1982)) it does not make easier the proof of his Theorem 4.5 using his method. In the zero mean case Engel's problem (the construction of an $L^2(\Omega)$-valued product stochastic measure) is more easily solved using Section 3.3, although the product stochastic measure obtained is not the same, as we shall see later. Multiple Wiener integrals have been applied to obtain expansions and stochastic integral representations of L^2-functionals of Y_t (Itô (1951), Kallianpur (1980)). Since the σ-fields generated by the processes $(X_1(t),\dots,X_n(t))$ and $(X_1(t)-m_1(t),\dots,X_k(t)-m_k(t))$ are the same, from the point of view of this application the zero mean assumption is unimportant. Moreover, this assumption enables us to use the techniques of Chapter II to construct integrals with respect to $\overset{n}{\underset{i=1}{\otimes}} X_i$ and identify the class of $\overset{n}{\underset{i=1}{\otimes}} X_i$-integrals that in this case is equal to $L^2(T^n, A^n, \overset{n}{\underset{i=1}{\otimes}} \mu_i)$. Engel (1982) does not consider stochastic integration w.r.t. his product stochastic measure $Y^{(n)}$.

Assumption (3.4.4) is used in Engel's work to assure that $Y^{(n)}$, as defined in (3.4.5), is an $L^2(\Omega)$-valued measure, i.e. Hilbert space valued.

In our framework, this additional moment condition is not necessary to obtain a (Hilbert space) $\overset{n}{\underset{i=1}{\otimes}} H_i$-valued measure $\overset{n}{\underset{i=1}{\otimes}} X_i$, where H is given in Proposition 3.3.1. Furthermore, for $n_1 \neq n_2$ the measures $\overset{n_1}{\underset{i=1}{\otimes}} X_i$ and $\overset{n_2}{\underset{i=1}{\otimes}} X_i$ take values in the same Hilbert space EXP(H) and they are orthogonal w.r.t. the inner product in EXP(H). Moreover, the identification of symmetric tensor products of H provides more insight into the structure of $\{X_1(t),\ldots,X_n(t)\}$. Such identification is known for important cases (Neveu (1968), Kallianpur (1970)), and we have obtained in Sections 3.2 and 3.3 identifications for other important situations.

As well as differences in the assumptions and in the techniques used, there are also important differences between the resulting product stochastic measures $Y^{(n)}$ and $\overset{n}{\underset{i=1}{\otimes}} X_i$. We now study some of these differences.

If $E \in A_o^n$ is as in (3.4.5), then

(3.4.7) $$Y^{(n)}(E) = \sum_{i_1\ldots i_n=1}^{q} c_{i_1\ldots i_n} X_1(I_{i_1})\ldots X(I_{i_n})$$

and

(3.4.8) $$\overset{n}{\underset{i=1}{\otimes}} X_i(E) = \sum_{i_1\ldots i_n=1}^{q} c_{i_1\ldots i_n} X_1(I_{i_1})\otimes\ldots\otimes X_n(I_{i_n}) \ .$$

Thus if $I_{j_1} <\ldots< I_{j_n}$, using Corollary 3.3.1 we obtain

(3.4.9) $$\overset{n}{\underset{i=1}{\otimes}} X_i(I_{j_1}\times\ldots\times I_{j_n}) = X_1(I_{j_1})\otimes\ldots\otimes X_n(I_{j_n})$$

$$= (n!)^{-\frac{1}{2}} X_1(I_{j_1})\ldots X_n(I_{j_n}) = (n!)^{-\frac{1}{2}} Y^{(n)}(I_{j_1}\times\ldots\times I_{j_n})$$

which suggests that $(n!)^{-\frac{1}{2}} Y^{(n)}$ and $\overset{n}{\underset{i=1}{\otimes}} X_i$ agree on antisymmetric sets (see Corollary 2.2.4), as it is shown by the following result.

<u>Proposition 3.4.1</u> For each permutation $\Pi = (\Pi_1,\ldots,\Pi_n)$ of $(1,\ldots,n)$ let

$$T^n_\Pi = \{(t_1,\ldots,t_n) \in T^n : t_{\Pi_1} < \ldots < t_{\Pi_n}\}$$

and

$$A^n_\Pi = A^n \cap T^n_\Pi .$$

Then

$$(3.4.10) \qquad \overset{n}{\underset{i=1}{\circledast}} X_i(A) = (n!)^{-\frac{1}{2}}\, Y^{(n)}(A) \qquad A \in A^n_\Pi .$$

Proof A^n_Π is the σ-field generated by the field of all elementary subsets of T^n_Π of the form

$$B = \bigcup_{1 \le i_{\Pi_1} < \ldots < i_{\Pi_n} \le q} c_{i_1 \ldots i_n} I_{i_1} \times \ldots \times I_{i_n}$$

where $I_1 < \ldots < I_q$ is a partition of intervals of T and $c_{i_1 \ldots i_n}$'s are as in (3.4.5). By (3.4.9) and the additivity property of the measures $Y^{(n)}$ and $\overset{n}{\underset{i=1}{\circledast}} X_i$

$$\overset{n}{\underset{i=1}{\circledast}} X_i(B) = (n!)^{-\frac{1}{2}}\, Y^{(n)}(B) \;.$$

Therefore from Theorem 3.3.2, since B is an antisymmetric set

$$(n!)\; E(Y^{(n)}(B))^2 = E(\overset{n}{\underset{i=1}{\circledast}} X_i(B))^2 = \frac{1}{n!} \quad \mu_1 \otimes \ldots \otimes \mu_n(B).$$

Then an approximation argument shows (3.4.10) for all $B \in A^n_\Pi$.

Q.E.D.

If A is not an antisymmetric set then (3.4.10) does not hold. Consider for example the case n=2, then from (3.4.7) and (3.4.8) if $B \in A^2_o$ is given by (3.4.5)

$$Y^{(2)}(B) = \sum_{i_1 i_2 = 1}^{q} c_{i_1 i_2} X_1(I_{i_1}) X_2(I_{i_2})$$

and using Theorem 3.3.3 and (3.3.44)

$$\overset{2}{\underset{i=1}{\circledast}} X_i(B) = \sum_{i_1,i_2=1}^{q} c_{i_1i_2} X_1(I_{i_1}) \circledast X_2(I_{i_2})$$

$$= (2!)^{-\frac{1}{2}} \{ \sum_{i_1i_2=1}^{q} c_{i_1i_2} \{X_1(I_{i_1}) X_2(I_{i_2}) - [X_1,X_2](I_{i_1} \cap I_{i_2})\}\}$$

$$= (2)^{-\frac{1}{2}} \{ \sum_{i_1i_2=1}^{q} c_{i_1i_2} X_1(I_{i_1}) X_2(I_{i_2}) - \sum_{i_1=1}^{q} c_{i_1i_1} [X_1,X_2](I_{i_1})\}$$

$$= (2)^{-\frac{1}{2}} \{Y^{(2)}(B) - \int_T 1_B(s,s) d[X_1,X_2]_s\}$$

i.e.

$$(3.4.11) \qquad Y^{(2)}(B) = (2)^{\frac{1}{2}} \overset{2}{\underset{i=1}{\circledast}} X_i(B) + \int_T 1_B(s,s) d[X_1,X_2]_s \qquad B \in A_o^n .$$

Then it follows from the above expression that even in the "zero mean" case we have that

$$E(Y^{(2)}(B)) = \int_T 1_B(s,s) d\mu_{12}(s) = \mu_{12}(B)$$

since $E(\overset{2}{\underset{i=1}{\circledast}} X_i(B)) = 0$ by Theorem 3.3.2 and

$$\mu_{12}(B) = E(X_1(B)X_2(B)) = E([X_1,X_2](B)).$$

Therefore Engel's product stochastic measure $Y^{(n)}$ is an uncentered measure which gives rise to an uncentered stochastic integral, while $\overset{n}{\underset{i=1}{\circledast}} X_i$ is centered.

Let $T = [0,1]$, $A = \mathcal{B}(T)$ and X be a single zero mean L^2-independently scattered measure on (T,A) with control measure μ. Rosinski and Szulga (1982) have considered the random product measure of X with itself in such a way that

$$(3.4.12) \qquad X^{(2)}(A_1 \times A_2) = X(A_1)\, X(A_2) \qquad A_1, A_2 \in A$$

can be extended to an $L^1(\Omega)$-valued vector measure on $(T \times T,\ A \times A)$. Under the additional assumption of $X(A) \in L^4(\Omega)$ all $A \in A$, they have shown that $X^{(2)}$

can be extended to an $L^2(\Omega)$-valued vector measure on $(T\times T,\ \mathcal{A}\times\mathcal{A})$. This last case corresponds to the Engel's situation n=2 and $X = X_1 = X_2$. They do not go beyond the case n=2. One of the complications that will appear is that for n=3, if $X(A)\in L^2(\Omega)$ $A\in\mathcal{A}$; then

$$X^{(3)}(A_1\times A_2\times A_3) = X(A_1)X(A_2)X(A_3) \qquad A_i\in\mathcal{A} \quad i=1,2,3$$

is not necessarily an element of $L^1(\Omega)$. This means more moment conditions about X are required and that product random measures of different orders take values in distinct spaces. Rosinski and Szulga (1982) use the theory of integration with respect to vector valued measures to construct integrals with respect to $X^{(2)}$, characterizing the class of $X^{(2)}$-integrable functions. For purposes of comparison some results of Rosinski and Szulga (1982) are summarized in the next two propositions.

<u>Proposition 3.4.2</u> (Rosinski and Szulga (1982)). Let $(T,\mathcal{A},\mu)$ be as above and X be a zero mean L^2-independently scattered measure on $(T,\mathcal{A})$ with control measure μ. For $A_1,A_2\in\mathcal{A}$ define $X^{(2)}(A_1\times A_2) = X(A_1)X(A_2)$. Then

a) $X^{(2)}$, as a vector measure in $L^1(\Omega)$, has a countably additive extension to $(T^2,\mathcal{A}^2)$.

b) For a real valued measurable function f on T^2 define

$$N(f) = \int_T |f(t,t)|d\mu(t) + \{\int\int_{T^2\setminus\Delta} |f(s,t)|^2 d\mu(s)d\mu(t)\}^{1/2}$$

where $\Delta = \{(s,t)\in T^2 : s=t\}$. If $N(f)<\infty$ then f is $X^{(2)}$-integrable with $L^1(\Omega)$-valued integral denoted by

$$\int_{T^2} f(s,t)dX^{(2)}(s,t).$$

c) If $N(f)<\infty$ then

(3.4.13) $$\int_{T^2} f(s,t)dX^{(2)}(s,t) = \int_T f(t,t)dV_1(t) + \int_{T^2} f(s,t)dV_2(s,t) \text{ a.s.}$$

where $V_1(t) = [X,X]_t$ is the optional quadratic variation process of $X(t) = X([0,t])$ and $V_2(A) = X^{(2)}(A\backslash\Delta) \quad A \in \mathcal{A}^2$.

Moreover, the first integral on the RHS of (3.4.13) belongs to $L^1(\Omega)$ while the second belongs to $L^2(\Omega)$.

d) Let X be a Gaussian random measure. Then f is $X^{(2)}$-integrable if and only if $N(f) < \infty$.

<u>Proposition 3.4.3</u> (Rosinski and Szulga (1982)). Let X be an independently scattered measure as in Proposition 3.4.2 and asuume that $E(X(A))^4) < \infty$ for all $A \in \mathcal{A}$. For $A_1, A_2 \in \mathcal{A}$ define $X^{(2)}(A_1 \times A_2) = X(A_1)X(A_2)$. Then

a) $X^{(2)}$, as a vector measure in $L^2(\Omega)$, has a countably additive extension to $(T^2, \mathcal{A}^2)$.

b) A real valued function on T^2 is $X^{(2)}$-integrable if and only if the next three conditions are satisfied.

(i) $\int_T |f(t,t)|d\mu(t) < \infty$

(ii) $\iint_{T^2\backslash\Delta} |f(t,s)|^2 d\mu(t)d\mu(s) < \infty$

(iii) $\int_T |f(t,t)|^2 |G|(dt) < \infty$

where $|G|$ is the variation of the signed measure

$$G(A) = E(X(A))^4 - 3(E(X(A))^2)^2 \qquad A \in \mathcal{A}.$$

Denote by

$$\int_{T^2} f(s,t)dX^{(2)}(s,t)$$

the $L^2(\Omega)$-valued integral of f w.r.t. $X^{(2)}$.

c) If f is $X^{(2)}$-integrable

$$E\left(\int_{T^2} f\,dX^{(2)}\right)^2 = \left(\int_T f(t,t)\,d\mu(t)\right)^2 + \int\int_{T^2} f^2(s,t)\,d\mu(s)\,d\mu(t)$$

$$+ \int\int_{T^2} f(s,t)f(t,s)\,d\mu(s)\,d\mu(t) + \int_T |f(t,t)|^2 G(dt).$$

d) If X is Gaussian then $G=0$ and the random integrals with respect to $X^{(2)}$ in the sense of L_1 and L_2 coincide.

On the other hand, following the notation of Proposition 2.2.1, let $X^{\circledast 2}$ be the $L^2(\Omega, F^X, P)$-valued symmetric tensor product stochastic measure on (T^2, A^2) constructed at the end of Section 3.3 (n=2) under the only moment assumption of X being L^2-valued. From Theorem 3.3.3 and (3.3.44) if $A_1, A_2 \in A$

$$X^{\circledast 2}(A_1 \times A_2) = X(A_1) \circledast X(A_2) = c(X(A_1)X(A_2) - [X,X](A_1 \cap A_2))$$

where $c = (2)^{-\frac{1}{2}}$, and therefore by (3.4.12)

$$X^{\circledast 2}(A_1 \times A_2) = c(X^{(2)}(A_1 \times A_2) - [X,X](A_1 \cap A_2)). \tag{3.4.14}$$

Hence since $E(X^{\circledast 2}(A_1 \times A_2)) = 0$

$$E(X^{(2)}(A_1 \times A_2)) = E([X,X](A_1 \cap A_2)) = \mu(A_1 \cap A_2)$$

i.e. $X^{(2)}$ is not necessarily a centered product random measure and gives rise to an uncentered integral as it is shown in (3.4.13) of Proposition 3.4.2. Moreover, from (3.4.14) and (3.4.13) we obtain that

$$c\,V_2(A) = X^{\circledast 2}(A) \qquad A \in A^2.$$

If $f \in L^2(T^2, A^2, \mu^{\circledast 2})$ then f is $X^{\circledast 2}$-integrable and the converse holds if μ is non-atomic. The integral with respect to $X^{\circledast 2}$ is centered (Proposi-

tion 3.3.3 taking n=2 and $X = X_1 = X_2$).

Then the advantages of the symmetric tensor product measure approach are that it does not need additional higher moment conditions to construct an $L^2(\Omega)$-valued random product measure and it gives rise to centered multiple stochastic integrals. Moreover, this approach can be used (as it was done in Sections 3.1, 3.2 and 3.3) to construct product stochastic measures of order $n \geq 1$, all taking values in the same space $L^2(\Omega, F^X, P)$, without needing extra higher moment conditions.

Concluding remarks We have seen in this chapter that the symmetric tensor product approach is an appropriate tool to obtain product stochastic measures and to construct multiple stochastic integrals w.r.t. them. A clear relationship between the theory of multiple stochastic integrals and the theory of vector valued measures has been established. Moreover, Theorem 2.1.4 suggests that this approach could be used to construct infinite product stochastic measures, which we have not done since we have not been succesful in defining the concept of infinite symmetric tensor product. This last notion was not found in the literature.

CHAPTER IV

NUCLEAR SPACE VALUED WIENER PROCESS AND STOCHASTIC INTEGRALS

In this chapter we bring together several notions and results about nuclear space valued Wiener processes that will be used in the next chapter. We begin by presenting the Countably Hilbert Nuclear Space Φ that we are going to consider in the remaining part of this work (Assumption 4.1.1). Then we define a Φ'-valued Wiener process (W_t) $t\geq 0$ with a continuous positive definite bilinear form Q on $\Phi\times\Phi$ and study some of its properties such as the corresponding Rigged Hilbert Space associated with it; the Wiener integral and its associated Gaussian space; and an infinite system of independently scattered measures that are non-identically distributed and mutually independent on disjoint sets. At the end of Section 4.1 we present some examples of Φ'-valued Wiener processes that show how our framework includes many cases already considered in the literature. In Section 4.2 we discuss real valued and Φ'-valued stochastic integrals with respect to W_t in a manner that they can be used in Chapter V in representing nonlinear functionals of W_t.

4.1 Nuclear space valued Wiener process

4.1.1 The Countably Hilbert Nuclear Space Φ and its n^{th} tensor product $\Phi^{\otimes n}$

Suppose E is a real linear space whose topology is determined by a countable family of Hilbertian semi-norms $\|\cdot\|_n$ $(\langle\cdot,\cdot\rangle_n)$ $n\geq 0$. For each n

let E_n be the Hilbert space completion of E with respect to $||\cdot||_n$. For $n<m$ suppose we have $||\phi||_n \leq ||\phi||_m$ $\phi \in E$. Then

$$E \subset E_m \subset E_n \subset E'$$

E' being the topological dual space of E. Furthermore, let $E = \bigcap_{n\geq 0} E_n$ and suppose that for every n there is an $m\geq n$ such that the injection of E_m into E_n is a Hilbert-Schmidt map. Then E is said to be a <u>Countably Hilbert Nuclear Space</u> (CHNS).

Among important properties of a CHNS E we have that (Gelfand and Vilenkin (1964)) E is a complete metrizable locally convex space (Frechet space) which is separable and every bounded closed set in E is compact. A useful result that we will use several times in this work is the following lemma. Although it can be proved for any linear topological space of the second category (see Xia (1972) page 386), we shall establish and prove it in the case when E is a CHNS. A result of this type, involving a Baire category argument, was first used in the study of E'-valued stochastic processes in Mitoma (1981a, 1981b).

<u>Lemma 4.1.1</u> Let E be a CHNS and let $V(\phi)$ be a non-negative, lower semicontinuous functional on E (i.e. $\phi_n \to \phi$ in E implies that $V(\phi) \leq \underline{\lim}\, V(\phi_n)$), satisfying the following conditions:

a) For any ϕ, $\psi \in E$ $\quad V(\phi+\psi) \leq V(\phi) + V(\psi)$.

b) For any $\phi \in E$ and $a \in \mathbb{R}$ $\quad V(a\phi) = |a| V(\phi)$.

c) $V(\phi) < \infty$ for any $\phi \in E$.

Then $V(\phi)$ is continuous on E and there exist a positive real number θ and a positive integer r such that

$$V(\phi) \leq \theta \, ||\phi||_r \qquad \forall\, \phi \in E.$$

<u>Proof</u> Let

$$D_n = \{\phi \in E: V(\phi) \le n\} .$$

Since V is a lower semicontinuous function on E then for each $n \ge 1$ D_n is a closed set of E (see Reed and Simon (1980)). Condition (c) implies that

$$E = \bigcup_{n=1}^{\infty} D_n .$$

Then by the Baire category theorem, since E is a complete metric space, it is never the union of a countable number of nowhere dense sets. Therefore there exists n_o such that D_{n_o} is not a nowhere dense set, i.e. there exist $\phi_o \in E$, $\delta_1 > 0$ and a positive integer r such that

$$\mathcal{U}_{\phi_o} = \{\phi \in E: ||\phi - \phi_o||_r < \delta_1\} \subset D_{n_o} .$$

Then for any $\phi \in E$ $\phi \ne 0$ if $\delta < \delta_1$

$$\delta \frac{\phi}{||\phi||_r} + \phi_o \in \mathcal{U}_{\phi_o} \quad \text{and} \quad \phi_o - \delta \frac{\phi}{||\phi||_r} \in \mathcal{U}_{\phi_o}$$

and hence they belong to D_{n_o}, i.e.

$$V(\delta \frac{\phi}{||\phi||_r} + \phi_o) \le n_o \text{ and } V(\phi_o - \delta \frac{\phi}{||\phi||_r}) \le n_o .$$

But using (b) with $a = -1$ $V(\delta \frac{\phi}{||\phi||_r} - \phi_o) = V(\phi_o - \frac{\delta\phi}{||\phi||_r}) \le n_o$.

Then by (a)

$$V(2\delta \frac{\phi}{||\phi||_r}) \le V(\delta \frac{\phi}{||\phi||_r} + \phi_o) + V(\delta \frac{\phi}{||\phi||} - \phi_o) \le 2n_o$$

and hence using (b), if $\theta = n_o/\delta$

$$V(\phi) \le \theta \, ||\phi||_r \qquad \forall \, \phi \in E.$$

Then the continuity of V follows since using (a) we obtain

$$|V(\phi) - V(\psi)| \le V(\phi-\psi) \le \theta \, \|\theta-\psi\|_r .$$

Q.E.D.

As examples of CHNS we have $S(\mathbb{R}^d)$ the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^d$ $d \ge 1$ and $S(\mathbb{Z}^d)$ the space of all rapidly decreasing sequences on $\mathbb{Z}^d$, the d-dimensional lattice space. Stochastic processes taking values in duals of these spaces have been considered in the recent works of K. Itô (1978a, 1978b, 1983), Dawson and Salehi (1980) and Shiga and Shimizu (1980) among others. However, in several practical problems, like those occurring in neurophysiology, it is not possible to fix in advance the space in which the stochastic processes take their values (see Kallianpur and Wolpert (1984)). The next example is taken from the work of the last named authors (see also Daletskii (1967)).

Example 4.1.1 (Kallianpur and Wolpert (1984)). Suppose a strongly continuous semigroup $(T_t)_{t\ge0}$ given on a Hilbert space H_o (that can be taken as $H_o = L^2(X, d\Gamma)$ for some σ-finite measure space (X,Σ,Γ)). The semigroup $(T_t)_{t\ge0}$ usually describes the evolutionary phenomenon being studied, such as the behavior of the voltage potential of a neuron (Kallianpur and Wolpert (1984)). Suppose that the strongly continuous and self adjoint semigroup $(T_t)_{t\ge0}$ satisfies the following two conditions:

(4.1.1) The resolvent $R_\alpha = \int_0^\infty e^{-\alpha t} T_t \, dt$ is compact for each $\alpha > 0$.

(4.1.2) For some $r_1 > 0$ $(R_\alpha)^{r_1}$ is a Hilbert-Schmidt operator.

By the Hille-Yosida theorem (T_t) has a negative definite infinitesimal generator $-L$. Then by Corollaries 4.4.1 and 4.4.2 in Balakrisnan (1981), H_o admits a complete orthonormal set $\{\phi_j\}_{j\ge1}$ of eigenvectors of L with

eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ satisfying

$$(4.1.3) \qquad \sum_{j=1}^{\infty} (\alpha+\lambda_j)^{-2r_1} < \infty \qquad (r_1 > 0).$$

Set

$$(4.1.4) \qquad \theta_1 = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} .$$

Denote by $\langle \cdot, \cdot \rangle_o$ the inner product in H_o and let

$$(4.1.5) \qquad \Phi = \{\phi \in H_o : \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_o^2 (1+\lambda_j)^{2r} < \infty \text{ for all } r \in \mathbb{R}\}.$$

For each $r \in \mathbb{R}$ define an inner product $\langle \cdot, \cdot \rangle_r$ and norm $\|\cdot\|_r$ on Φ by

$$(4.1.6) \qquad \langle \phi, \psi \rangle_r = \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_o \langle \psi, \phi_j \rangle_o (1+\lambda_j)^{2r}$$

$$(4.1.7) \qquad \|\phi\|_r^2 = \langle \phi, \phi \rangle_r$$

and let H_r be the Hilbert space completion of Φ in the inner product $\langle \cdot, \cdot \rangle_r$. Then Φ with the Frechet topology determined by the family $\{\|\cdot\|_r\}_{r \in \mathbb{R}}$ of Hilbertian norms is a Countably Hilbert Nuclear Space. Let $\Phi' = \bigcup_r H_r$ with the inductive limit topology. Then Φ' is identified with the dual space (in the weak topology) to Φ. The following properties hold (see Kallianpur and Wolpert (1984)):

(4.1.8) H_{-r} and H_r are in duality under the pairing

$$\xi[\phi] = \sum_{j=1}^{\infty} \langle \xi, \phi_j \rangle_{-r} \quad \xi \in H_{-r}, \quad \phi \in H_r .$$

(4.1.9) $\Phi \subset H_s \subset H_r \subset \Phi'$ if $r < s$ and the injection of H_s into H_r is a Hilbert-Schmidt map if $s > r + r_1$.

(4.1.10) Finite linear combinations of $\{\phi_j\}$ are dense in Φ and in every H_r; moreover, $\{\phi_j\}_{j \geq 1}$ is an orthogonal system in

each H_r, and then $\{(1+\lambda_j)^{-r}\phi_j\}_{j\geq 1}$ is a CONS for H_r.

Assumption 4.1.1 From now on, unless explicitly stated otherwise, we will assume that Φ is the CHNS of (4.1.5) and that L, $\{\lambda_j\}_{j\geq 1}$, $\{\phi_j\}_{j\geq 1}$, r_1, θ_1, H_r $-\infty < r < \infty$, and Φ' are as in the last example.

We will see in Examples 4.1.2 and 4.1.3 how the CHN spaces $S(\mathbb{R}^d)$ and $S(\mathbb{Z}^d)$ may be obtained within this framework.

The tensor product nuclear space $\Phi^{\otimes n}$ Under the notation and the hypotheses of Example 4.1.1, for each $r \in \mathbb{R}$ let $H_r^{\otimes n}$ be the n-fold tensor product Hilbert space of H_r with inner product $\langle\cdot,\cdot\rangle_{r^{\otimes n}}$ and norm $\|\cdot\|_{r^{\otimes n}}$.

Define the linear space

$$\Phi^{\otimes n} = \{\psi \in H_0^{\otimes n} : \psi \in H_r^{\otimes n} \text{ all } r \in \mathbb{R}\}$$

with the topology determined by the family of norms $\{\|\cdot\|_{r^{\otimes n}}\}_{r\in\mathbb{R}}$.

Proposition 4.1.1 For each $n \geq 1$ $(\Phi^{\otimes n}, \|\cdot\|_{r^{\otimes n}}\ r \in \mathbb{R})$ is a Countably Hilbert Nuclear Space. It is called the n-fold tensor product nuclear space of Φ. If $r < s$ then

$$\Phi^{\otimes n} \subset H_s^{\otimes n} \subset H_r^{\otimes n} \subset (\Phi^{\otimes n})'$$

where $(\Phi^{\otimes n})'$ is the inductive limit of $H_r^{\otimes n}$ $r \in \mathbb{R}$, identified with the dual to $\Phi^{\otimes n}$ in the weak topology.

Proof Since for $r < s$ $H_s \subset H_r$ and $\|\ \|_r \leq \|\ \|_s$, then for each $n \geq 1$ $H_s^{\otimes n} \subset H_r^{\otimes n}$ and $\|\ \|_{r^{\otimes n}} \leq \|\ \|_{s^{\otimes n}}$. Then

(4.1.11) $$\Phi^{\otimes n} = \bigcap_{k=0}^{\infty} H_k^{\otimes n}$$

and therefore $\Phi^{\otimes n}$ is a complete metric space.

Thus we only have to prove that if $s > r + r_1$, the injection from $H_s^{\otimes n}$ into $H_r^{\otimes n}$ is a Hilbert-Schmidt map. Let $\{e_j = (1+\lambda_j)^{-s}\phi_j\}_{j\geq 1}$ be a CONS for H_s, then $\{e_{j_1}\otimes\cdots\otimes e_{j_n}\}_{j_1,\ldots,j_n\geq 1}$ is a CONS for $H_s^{\otimes n}$ and using (4.1.10) and (4.1.6)

$$\sum_{j_1\ldots j_n=1}^{\infty} \| e_{j_1}\otimes\cdots\otimes e_{j_n}\|^2_{r^{\otimes n}} = \sum_{j_1\ldots j_n=1}^{\infty} \| e_{j_1}\|^2_r \cdots \| e_{j_n}\|^2_r$$

$$= \sum_{j_1\ldots j_n=1}^{\infty} (1+\lambda_{j_1})^{-2s}\cdots(1+\lambda_{j_n})^{-2s} \|\phi_{j_1}\|^2_r\cdots\|\phi_{j_n}\|^2_r$$

$$= \sum_{j_1\ldots j_n=1}^{\infty} \Big(\prod_{i=1}^{n}(1+\lambda_{j_i})^{-2s}\Big)\Big(\prod_{i=1}^{n}(1+\lambda_{j_i})^{2r}\Big) = \Big(\sum_{j=1}^{\infty}(1+\lambda_j)^{-2(s-r)}\Big)^n \leq \theta_1^n < \infty .$$

Thus the injection from $H_s^{\otimes n}$ into $H_r^{\otimes n}$ is a Hilbert-Schmidt map.

Q.E.D.

The next proposition will be useful in studying multiple Wiener integrals. It gives the n^{th} tensor quadratic form of a continuous positive definite bilinear (c.p.d.b.) form on $\Phi\times\Phi$.

<u>Proposition 4.1.2</u> Let $Q(\cdot,\cdot)$ be a continuous positive definite bilinear form on $\Phi\times\Phi$. Define

$$(4.1.12) \qquad Q^{\otimes n}(\psi_1\otimes\cdots\otimes\psi_n,\ \eta_1\otimes\cdots\otimes\eta_n) = Q(\psi_1,\eta_1)\cdots Q(\psi_n,\eta_n) \qquad \psi_i,\eta_i\in\Phi .$$

Then $Q^{\otimes n}$ can be extended to a continuous positive definite bilinear form on $\Phi^{\otimes n}\times\Phi^{\otimes n}$. Moreover, there exist $\theta_2 > 0$ and $r_2 > 0$ such that

$$(4.1.13) \qquad Q^{\otimes n}(\eta,\eta) \leq \theta_2^n \|\eta\|^2_{r_2^{\otimes n}} \qquad \eta\in\Phi^{\otimes n} .$$

<u>Proof</u> The first part follows as in the construction of a tensor product inner product (see Proposition 1, page 49 of Reed and Simon (1980)). To

prove (4.1.13), since Q is a c.p.d.b. form on $\Phi\times\Phi$, by the nuclear theorem there exist $\theta_2 > 0$ and $r_2 > 0$ such that

$$Q(\phi,\phi) \le \theta_2 \, ||\phi||^2_{r_2} \qquad \phi\in\Phi \ .$$

Then using (4.1.12), for $\psi_i \in \Phi \quad i=1,\dots,n$

$$Q^{\otimes n}(\psi_1\otimes\dots\otimes\psi_n,\ \psi_1\otimes\dots\otimes\psi_n) \le \theta_2^n \ ||\psi_1\otimes\dots\otimes\psi_n||^2_{r_2^{\otimes n}}$$

and (4.1.13) follows from the extension of $Q^{\otimes n}$.

Q.E.D.

We finish this section by showing how the nuclear spaces $S(\mathbb{R}^d)$ and $S(\mathbb{Z}^d)$ may be obtained in the framework of Example 4.1.1.

Example 4.1.2 $(S(\mathbb{R}^d))$ (Itô (1978a)). Let $H_0 = L^2(\mathbb{R},m)$, m Lebesgue measure in $\mathbb{R}$, $-L = d^2/dx^2 - x^2/4$ be the harmonic oscillator (Reed and Simon (1980)), $L\phi_n = \lambda_n\phi_n \quad n=1,2,\dots$ where $\{\phi_n\}_{n\ge 1}$ are Hermite functions, $\lambda_n = n-\frac{1}{2}$ $n\ge 1$

$$\phi_{k+1}(x) = (g(x))^{\frac{1}{2}}\, h_k(2^{-\frac{1}{2}}x)(2^k k!(\pi)^{-\frac{1}{2}})^{-\frac{1}{2}} \qquad k\ge 0$$

$g(x) = (2\pi)^{-\frac{1}{2}}\, e^{-x^2/2}$ and h_k are Hermite polynomials defined by

$$h_k(x) = {}^{-k/2}(-1)^k e^{x^2}\, \frac{d^k\,(e^{-x^2})}{dx^k} \qquad k=0,1,2,\dots\ .$$

Then (see Itô (1978a)) $r_1 = 1$, $\theta_1 = \pi/2$ and Φ as defined in Example 4.1.1 is the space $S(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$, with the topology defined by the family of Hilbertian norms

$$||\phi||^2_p = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})^{2p} <\phi,\phi_n>_0^2 \quad p\ge 0,\quad <\phi,\phi_n>_0 = \int_{\mathbb{R}} \phi(x)\phi_n(x)\,dx\ .$$

In a similar fashion Itô (1978a) constructs the space $S(\mathbb{R}^d)$ $d\ge 1$ which can be seen as the d-fold tensor product nuclear space $(S(\mathbb{R}))^{\otimes d}$.

Example 4.1.3 ($S(\mathbb{Z}^d)$). Let $H_0 = \ell_2$ be the Hilbert space of all real sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$, $Lx = \{nx_n\}$. Then $\lambda_n = n$ and $\phi_n = e_n$ $n \geq 1$, where $\{e_n\}_{n\geq 1}$ is the canonical basis in ℓ_2. Thus $r_1 = 1$, $\theta_1 = \pi/6$ and the space Φ defined by

$$\Phi = \{x \in \mathbb{R}^{\infty}: \sum_{n=1}^{\infty} (n+1)^{2P} x_n < \infty \quad \text{all } P \geq 0\}$$

and topologized by the family of Hilbertian norms

$$||x||_p^2 = \sum_{n=1}^{\infty} (n+1)^{2P} x_n^2 \qquad P \geq 0$$

is the space $S(\mathbb{Z})$ of all rapidly decreasing sequences. The space $S(\mathbb{Z}^d)$ may be constructed as the d-fold tensor product nuclear space of $S(\mathbb{Z})$.

4.1.2 Φ'-valued Wiener process

Throughout this section we assume the hypotheses and notation of Example 4.1.1. Let (Ω, F, P) be a fixed but arbitrary complete probability space. All Φ'-valued random elements and Φ'-valued stochastic processes considered in this section are defined on this probability space. We denote by $\mathcal{B}(\Phi')$ the σ-field on Φ' generated by the sets

$$E_{\phi,a} = \{\xi \in \Phi': \xi(\phi) < a\} \qquad a \in \mathbb{R}, \ \phi \in \Phi$$

which is the σ-field generated by the open sets in the weak topology. Measures on Φ', Φ'-valued random elements and Φ'-valued stochastic processes are defined with respect to the σ-field $\mathcal{B}(\Phi')$. Thus a mapping

$$X_t(\omega): [0,\infty) \times \Omega \to \Phi'$$

is a Φ'-valued stochastic process if and only if $X_t(\cdot)[\phi]$ is a real valued stochastic process for all $\phi \in \Phi$.

A Φ'-valued stochastic process (X_t) $t \geq 0$ is called an H_{-r}-valued process if for every $t \geq 0$ X_t is an H_{-r}-valued element.

From now on we will write $\mathbb{R}_+ = [0,\infty)$ and $A = \mathcal{B}(\mathbb{R}_+)$ will denote its Borel sets.

Definition 4.1.1 A sample continuous Φ'-valued stochastic process $W = (W_t)_{t\in\mathbb{R}_+}$ defined on (Ω,F,P) is called a (centered) Φ'-valued Wiener process with covariance $Q(\cdot,\cdot)$ if

a) $W_o \equiv \underline{0}$.

b) W_t has independent increments.

c) For each $\phi \in \Phi$ and $t \geq 0$

$$E(e^{iW_t[\phi]}) = \exp(-t/2\ Q(\phi,\phi))$$

where $Q(\cdot,\cdot)$ is a continuous positive definite bilinear (c.p.d.b.) form on $\Phi \times \Phi$.

From the above definition we see that the system

$$\{W_t[\phi];\ \phi \in \Phi, t \geq 0\}$$

is a Gaussian system of random variables and that if ϕ, $\psi \in \Phi$, the real valued processes $W_t[\phi]$ and $W_t[\psi]$ are independent on non-overlapping increments. Moreover, for each ϕ, $\psi \in \Phi$ and s, $t \in \mathbb{R}_+$

$$E(W_s[\phi]W_t[\psi]) = \min(s,t)\ Q(\phi,\psi).$$

If $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$, following Itô (1978a), W_t may be called a standard Φ'-valued Wiener process. If $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_r$ for some $r \in \mathbb{R}$, then $\{\phi_j\}_{j\geq 1}$, the system of eigenvectors of the generator L, diagonalize Q (see (4.1.10)). In general we will not assume that Q is diagonalized by the system $\{\phi_j\}_{j\geq 1}$.

Several examples of Φ'-valued Wiener processes with different covariances Q are presented in Section 4.1.3.

The existence of a Φ'-valued Wiener process with an H_{-q} continuous version is now established.

Theorem 4.1.1 Let $\{Y(t,\phi):\phi\in\Phi,\ t\in\mathbb{R}_+\}$ be a centered Gaussian system of random variables such that

$$E(Y(t,\phi)Y(s,\psi)) = \min(s,t)Q(\phi,\psi) \qquad \phi,\ \psi\in\Phi,\ s,\ t\in\mathbb{R}_+$$

where Q is a c.p.d.b. form on $\Phi\times\Phi$. Then there exists a Φ'-valued Wiener process (W_t) $t\in\mathbb{R}_+$ with c.p.d.b. form Q such that $Y(t,\phi)=W_t[\phi]$ a.s. for all $\phi\in\Phi$, $t\in\mathbb{R}_+$, and W_t has an H_{-q}-valued continuous version for some $q\geq r_1+r_2$, where r_2 is such that for some $\theta_2>0$

$$Q(\phi,\phi) \leq \theta_2\|\phi\|^2_{r_2} \qquad \forall\ \phi\in\Phi\ . \tag{4.1.14}$$

Proof Using the bilinearity of Q, for $t\geq 0$ fixed and $c_1,c_2\in\mathbb{R}$, $\phi_1,\phi_2\in\Phi$ we obtain that

$$E(Y(t,c_1\phi_1+c_2\phi_2) - Y(t,c_1\phi_1) - Y(t,c_2\phi_2))^2 = 0\ .$$

By the Kernel theorem for CHNS's (Gelfand and Vilenkin (1964)), there exist $\theta_2>0$ and an integer $r_2>0$ such that (4.1.14) is satisfied. Therefore

$$E|Y(t,\phi)|^2 \leq \theta_2 t\|\phi\|^2_{r_2} \qquad \phi\in\Phi\ . \tag{4.1.15}$$

Fix t throughout the argument. Then Y_t: $\phi\to Y(t,\phi)$ is a bounded linear operator from the pre-Hilbert space $(\Phi,\|\cdot\|_{r_2})$ into $L^2=L^2(\Omega,F,P)$ and hence extends uniquely to a bounded linear operator from H_{r_2} into L^2, denoted also by Y_t.

Let $\{(1+\lambda_j)^{-(r_1+r_2)}\phi_j\}_{j\geq 1}$ be a CONS for $H_{r_1+r_2}$. Write $\tilde{\phi}_j=(1+\lambda_j)^{-(r_1+r_2)}\phi_j$

$j \geq 1$ and let $\{f_j\}_{j\geq 1}$ be a CONS for $H_{-(r_1+r_2)}$ dual to $\{\tilde{\phi}_j\}_{j\geq 1}$, i.e. $\langle f_k, \tilde{\phi}_j \rangle_{-(r_1+r_2)} = \delta_{kj}$. Set $X_t^j = Y(t, \tilde{\phi}_j)$. Then from (4.1.15)

$$E\left(\sum_{j=1}^{\infty} (X_t^j)^2\right) \leq \theta_2 t \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|_{r_2}^2 = \theta_2 t \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} = \theta_1\theta_2 t > 0.$$

Next let $\Omega_1 = \{\omega \in \Omega: \sum_{j=1}^{\infty} (X_t^j(\omega))^2 < \infty\}$, then $P(\Omega_1) = 1$. Define

$$W_t(\omega) = \begin{cases} 0 & \omega \notin \Omega_1 \\ \sum_{j=1}^{\infty} X_t^j(\omega) f_j & \omega \in \Omega_1 \end{cases}.$$

Then for each $t \geq 0$ $W_t(\omega) \in H_q' = H_{-q}$ a.s. for $q \geq r_1 + r_2$ and

$$E\|W_t\|_{-q}^2 = E\left(\sum_{j=1}^{\infty} (X_t^j)^2\right) \leq \theta_1\theta_2 t.$$

Next for $\phi \in H_q$

$$W_t[\phi] = \sum_{j=1}^{\infty} X_t^j(\omega) f_j[\phi] = \sum_{j=1}^{\infty} X_t^j \langle \phi, \tilde{\phi}_j \rangle_q \tag{4.1.16}$$

and for $0 \leq s \leq t$

$$W_t[\phi] - W_s[\phi] = \sum_{j=1}^{\infty} (X_t^j - X_s^j) \langle \phi, \tilde{\phi}_j \rangle_q .$$

But
$$E(X_t^j - X_s^j)^2 = (t-s) Q(\tilde{\phi}_j, \tilde{\phi}_j) = (t-s)(1+\lambda_j)^{-2(r_1+r_2)} Q(\phi_j, \phi_j)$$
$$\leq (t-s)(1+\lambda_j)^{-2(r_1+r_2)} \theta_2 \|\phi_j\|_{r_2}^2 = \theta_2 (t-s)(1+\lambda_j)^{-2r_1} .$$

Then

$$E\|W_t - W_s\|_{-q}^2 = E \sup_{\|\phi\|_q \leq 1} |W_t[\phi] - W_s[\phi]|^2$$

$$= E \sup_{\|\phi\|_p \leq 1} \left(\sum_{j=1}^{\infty} (X_t - X_s)^2 \langle \phi, \tilde{\phi}_j \rangle_q\right)^2 \leq E \sup_{\|\phi\|_q \leq 1} \left(\sum_{j=1}^{\infty} (X_t^j - X_s^j)^2\right)\left(\sum_{j=1}^{\infty} \langle \phi, \tilde{\phi}_j \rangle_q^2\right)$$

$$\leq E\left(\sum_{j=1}^{\infty} (X_t^j - X_s^j)^2\right) \leq \theta_2 (t-s) \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} = \theta_2\theta_1 (t-s) < \infty .$$

Thus

$$(4.1.17) \qquad E\|W_t - W_s\|_{-q}^2 \le \theta_1\theta_2|t-s|$$

and applying Kolmogorov continuity theorem we have that there exists an H_{-q}-valued continuous version of (W_t) $t \in \mathbb{R}_+$ for $q \ge r_1 + r_2$.

Q.E.D.

Corollary 4.1.1 If (W_t) $t \in \mathbb{R}_+$ is a standard Φ'-valued Wiener process, then it has an H_{-r_1}-valued continuous version, where r_1 is as in (4.1.4).

Proof Since for a standard ϕ'-valued Wiener process $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_0$, then we have equality in (4.1.14) with $\theta_2 = 1$ and $r_2 = 0$.

Q.E.D.

If (W_t) $t \ge 0$ is a Φ'-valued Wiener process with c.p.d.b. form Q on $\Phi \times \Phi$, then $W(t,\phi) = W_t[\phi]$ is a centered Gaussian system and therefore by the last theorem W_t has an H_{-q}-valued continuous version, also denoted by W_t, for $q \ge r_1 + r_2$.

Assumption 4.1.2 From now on we will assume that (W_t) $t \in \mathbb{R}_+$ is a Φ'-valued Wiener process with c.p.d.b. form Q on $\Phi \times \Phi$ and an H_{-q} continuous version for $q \ge r_1 + r_2$, given by Theorem 4.1.1, where r_2 and θ_2 are as in (4.1.4). Moreover, assume that for each $t \ge 0$ $F_t^W = \sigma(W_s[\phi] : 0 \le s \le t,\ \phi \in \Phi)$ with F_0 containing all P-null sets of F.

Lemma 4.1.2 Let $q \ge r_1 + r_2$. Then for each $\phi \in H_q$ $(W_t[\phi], F_t^W)$ is a continuous martingale with quadratic variation process

$$(4.1.18) \qquad \langle W[\phi]\rangle_t = t\,Q(\phi,\phi) \qquad t \ge 0.$$

Moreover, the cross predictable quadratic variation of $W_t[\phi]$ and $W_t[\psi]$ for $\phi, \psi \in H_q$ is

$$\text{(4.1.19)} \qquad \langle W[\phi], W[\psi]\rangle_t = tQ(\phi,\psi) \qquad t \geq 0.$$

Proof The martingale property follows since $(W_t)_{t\geq 0}$ is a Φ'-valued process with independent increments and for $\phi \in H_q$ and $t \geq 0$ $E(W_t[\phi]) = 0$.

Next since for each $t \geq 0$

$$\langle W[\phi], W[\psi]\rangle_t = \frac{1}{2}\{\langle W[\phi]+W[\psi], W[\phi]+W[\psi]\rangle_t$$

$$- \langle W[\phi]\rangle_t - \langle W[\psi]\rangle_t\}$$

then we only have to prove (4.1.18). But from (4.1.16) since Q has a continuous extension to $H_q \times H_q$ for $q \geq r_1 + r_2$ (see Proposition 4.1.3) if $\phi \in H_q$ then

$$E(W_t[\phi])^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle \phi, \tilde{\phi}_j\rangle_q \langle \phi, \tilde{\phi}_k\rangle_q \, E(X_t^j X_t^k)$$

$$= t \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle \phi, \tilde{\phi}_j\rangle_q \langle \phi, \tilde{\phi}_k\rangle_q \, Q(\tilde{\phi}_j, \tilde{\phi}_k) = tQ(\phi,\phi)$$

and hence $E((W_t[\phi] - W_s[\phi])^2 | F_s^W) = (t-s)Q(\phi,\phi) \quad s < t \quad \phi \in \Phi$.

For $\phi \in H_q$ $q \geq r_1 + r_2$ $(W_t[\phi], F_t^W)$ has a continuous version since from Theorem 4.1.1 W_t has an H_{-q}-continuous version.

Q.E.D.

Corollary 4.1.2 Let $(W_t)_{t\geq 0}$ be a Φ'-valued Wiener process with a c.p.d.b. form Q on $\Phi \times \Phi$. Then if $q \geq r_1 + r_2$

$$E(W_t[\phi]W_s[\psi]) = \min(s,t)Q(\phi,\psi) \qquad \phi, \psi \in H_q$$

where $Q(\phi,\psi)$ $\phi, \psi \in H_q$ is a c.p.d.b. form on $H_q \times H_q$ that extends the c.p.d.b. form Q on $\Phi \times \Phi$.

Proof It follows from the proof of the last lemma by observing that if $s < t$

$$E(W_t[\phi]W_s[\psi]) = E(W_s[\psi]E(W_t[\phi]|F_s^W)) = E(W_s[\phi]W_s[\psi])$$

$$= \frac{1}{2}\{E(W_s[\phi+\psi])^2 - E(W_s[\phi])^2 - E(W_s[\psi])^2\}.$$

Q.E.D.

We now consider several concepts associated with a Φ'-valued Wiener process (W_t) $t \geq 0$ with a c.p.d.b. form Q on $\Phi \times \Phi$ and an H_{-q} continuous version for $q \geq r_1 + r_2$.

Rigged Hilbert space associated with a Φ'-valued Wiener process Let (W_t) $t \geq 0$ be a Φ'-valued Wiener process with a c.p.d.b. form Q on $\Phi \times \Phi$. Then Q is an inner product on $\Phi \times \Phi$. Denote by H_Q the completion of Φ with respect to Q and by $<,\cdot,>_Q$ or $Q(\cdot,\cdot)$ $(||\cdot||_Q)$ the corresponding inner product (norm) on H_q. Then

$$\Phi \subset H_s \subset H_Q \equiv H_Q' \subset H_{-s} \subset \Phi' \qquad s \geq r_2 \tag{4.1.20}$$

The system (4.1.20) is called the Rigged Hilbert Space (see Gelfand and Vilenkin (1964)) associated with the Φ'-valued Wiener process (W_t) $t \geq 0$ with c.p.d.b. form Q on $\Phi \times \Phi$.

Proposition 4.1.3 a) For $s \geq r_2$ and $\phi \in H_s$

$$Q(\phi,\phi) \leq \theta_2 ||\phi||_s^2 \tag{4.1.21}$$

and therefore (4.1.20) makes sense, and Q has a continuous extension to $H_s \times H_s$.

b) For $s \geq r_1 + r_2$ the injection of H_s into H_Q is a Hilbert-Schmidt map.

Proof a) From the Kernel theorem (see (4.1.14))

$$Q(\phi,\phi) \leq \theta_2 ||\phi||_{r_2}^2 \qquad \forall\ \phi \in \Phi$$

Next if $\psi \in H_s$ since Φ is dense in H_s there exists a sequence $\{\psi_n\}$ in Φ such that $\psi_n \to \psi$ in H_s. Then $Q(\psi_n-\psi_m,\psi_n-\psi_m) \le \theta_2 \|\psi_n-\psi_m\|_s^2 \to 0$ which implies that $Q(\psi_n,\psi_n) \to Q(\psi,\psi)$ $n\to\infty$ and then (4.1.21) follows.

b) Let $\{\tilde{\phi}_j = (1+\lambda_j)^{-s}\phi_j\}_{j\ge 1}$ be a CONS for H_s, then

$$\sum_{j=1}^{\infty} Q(\tilde{\phi}_j,\tilde{\phi}_j) = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2s} Q(\phi_j,\phi_j) \le \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2s} \|\phi_j\|_{r_2}^2$$

$$= \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(s-r_2)} \le \theta_2\theta_1 < \infty$$

and then the injection of H_s into H_Q is a *Hilbert-Schmidt* map for $s \ge r_1 + r_2$.

Q.E.D.

In a similar way, for each $n \ge 1$ the Rigged Hilbert Space

$$\Phi^{\otimes n} \subset H_s^{\otimes n} \subset H_Q^{\otimes n} = (H_Q^{\otimes n})' \subset (\Phi^{\otimes n})'$$

may be constructed where

$$Q^{\otimes n}(\eta,\eta) \le \theta_2^n \|\eta\|_{s^{\otimes n}}^2 \qquad \eta \in H_s^{\otimes n} \qquad s \ge r_2$$

and the injection from $H_s^{\otimes n}$ into $H_Q^{\otimes n}$ is a Hilbert-Schmidt map for $s \ge r_1 + r_2$.

<u>Wiener integral and the Gaussian space $\mathcal{H}$ of W_t</u> Let $C = C([0,\infty) \to \Phi)$ be the linear manifold of all measurable step functions on $\mathbb{R}_+$ with values in Φ, i.e. $f \in C$ iff there exists a finite collection of positive real numbers $0 = t_0 < t_1 < t_2 < \ldots < t_k$ and $\alpha_i \in \Phi$ $i=1,\ldots,k$ such that

(4.1.22) $$f(t) = \sum_{i=1}^{k} \alpha_i \, 1_{(t_{i-1},t_i]}(t).$$

For $f \in C$ define the <u>Wiener integral</u>

(4.1.23) $$I_1(f) = \sum_{i=1}^{k} (W_{t_i} - W_{t_{i-1}})[\alpha_i].$$

Then $I_1(\cdot)$ has the following properties.

Lemma 4.1.3 Let $f, g \in C$. Then

a) $I_1(cf) = cI_1(f) \qquad c \in \mathbb{R}$.

b) $I_1(f+g) = I_1(f) + I_1(g)$.

c) $E(I_1(f)) = 0$.

d) $E(I_1(f)I_1(g)) = \int_{\mathbb{R}_+} Q(f(t),g(t))dt$.

e) $E(I_1(f))^2 = \int_{\mathbb{R}_+} ||f(t)||_Q^2 \, dt$.

Proof (a) and (b) are proved as in the real valued case using the fact that for each $t \geq 0$ $W_t \in \Phi'$. The proof of (c) follows since $E(W_t[\phi]) = 0$ $\forall\ \phi \in \Phi$ and $t \geq 0$. To prove (d) write $A_i = (t_{i-1}, t_i]$ and $W(A_i) = W_{t_i} - W_{t_{i-1}}$. Then if

$$f(t) = \sum_{i=1}^{k} \alpha_i \, 1_{A_i}(t)$$

and

$$g(t) = \sum_{i=1}^{k} \beta_i \, 1_{A_i}(t)$$

for $0 \leq t_o < t_1 < \ldots < t_k$, α_i, $\beta_i \in \Phi$ $i=1,\ldots,k$, by definition of I_1

$$I_1(f) = \sum_{i=1}^{k} W(A_i)[\alpha_i]$$

$$I_1(g) = \sum_{i=1}^{k} W(B_i)[\alpha_i] \cdot$$

Them if $m(\cdot)$ denotes the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$

$$E(I_1(f)I_1(g)) = \sum_{i=1}^{k} \sum_{j=1}^{k} E(W(A_i)[\alpha_i]W(A_j)[\beta_j])$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} m(A_i \cap A_j) Q(\alpha_i, \beta_j) = \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathbb{R}_+} 1_{A_i \cap A_j}(t) Q(\alpha_i, \beta_j) dt$$

$$= \int_{\mathbb{R}_+} Q\left(\sum_{i=1}^{k} \alpha_i 1_{A_i}(t), \sum_{j=1}^{k} \beta_j 1_{A_j}(t)\right)dt = \int_{\mathbb{R}_+} Q(f(t),g(t))dt$$

which proves (d) and (e). Q.E.D.

To extend I_1 to $L^2(\mathbb{R}_+ \to H_Q)$ we first prove the following result, where $L^2(\mathbb{R}_+ \to H_Q)$ is the Hilbert space of H_Q-valued measurable functions f on $\mathbb{R}_+$ (identifying those which are equal a.e. dt) such that

$$\int_{\mathbb{R}_+} \|f(t)\|_Q^2 \, dt < \infty .$$

Lemma 4.1.4 C is a dense linear manifold in $L^2(\mathbb{R}_+ \to H_Q)$.

Proof Let $f \in L^2(\mathbb{R}_+ \to H_Q)$, then for each $\varepsilon > 0$ there exists an H_Q-valued step function f^ε, i.e.

$$f^\varepsilon(t) = \sum_{i=1}^{k} a_i 1_{A_i}(t)$$

$a_i \in H_Q$, $A_i = (t_{i-1}, t_i]$ $i=1,\ldots,k$ $0 \le t_o < t_1 < \ldots < t_k$ $k \ge 1$, such that

$$\int_{\mathbb{R}_+} \|f(t) - f^\varepsilon(t)\|_Q^2 \, dt < \varepsilon/2. \tag{4.1.24}$$

Since Φ is dense in H_Q, there exist $\alpha_i \in \Phi$ $i=1,\ldots,k$ such that

$$\|a_i - \alpha_i\|_Q^2 < \frac{\varepsilon}{2k(t_i - t_{i-1})} \qquad i=1,\ldots,k.$$

Define

$$g^\varepsilon(t) = \sum_{i=1}^{k} \alpha_i 1_{A_i}(t)$$

then $g^\varepsilon \in C$ and

$$\int_{\mathbb{R}_+} \|f^\varepsilon(t) - g^\varepsilon(t)\|_Q^2 \, dt < \varepsilon/2 .$$

Then from the last expression and (4.1.24)

$$\int_{\mathbb{R}_+} \|f(t) - g^\varepsilon(t)\|_Q^2 \, dt < \varepsilon$$

i.e. C is a dense linear manifold in $L^2(\mathbb{R}_+ \to H_Q) \cong L^2(\mathbb{R}_+) \otimes H_Q$.

Q.E.D.

The Gaussian space (linear space) associated with the Φ'-valued Wiener process W_t is defined as

(4.1.25) $$H = L_1(W) = \overline{sp}\{W_t[\phi]: \phi \in \Phi,\ t \geq 0\}$$

where the closure is taken with respect to $L^2(\Omega, F^W, P)$, where $F^W = F^W_\infty$. From Proposition 7.3. in Neveu (1968)

(4.1.26) $$L^2(\Omega, F^W, P) \overset{\eta_1}{\cong} \sum_{n \geq 0} \oplus H^{\odot n}$$

where

$$\eta_1(\exp \odot(h)) = \exp(h - E(h^2)) \quad h \in H \text{ and}$$

$$\exp \odot(h) = (1, h, \frac{1}{\sqrt{2!}} h^{\odot 2}, \frac{1}{\sqrt{3!}} h^{\odot 3}, \ldots)$$

as it was shown in Chapter III (see also Kallianpur (1980) Chapter VI). Then for all $n \geq 0$ $H^{\odot n}$ may be seen as a closed subspace of $L^2(\Omega, F^W, P)$. This fact will be used in Sections 5.1 and 5.2 together with the next definition.

Definition 4.1.2 Lemma 4.1.3 shows that I_1 is an isometry from the linear space C of Φ'-valued step functions into H. Hence from Lemma 4.1.4 this isometry can be extended uniquely to an isometry from $L^2(\mathbb{R}_+ \to H_Q)$ onto H, also denoted by I_1 and called the Wiener integral. It has the properties (a)-(e) of Lemma 4.1.3.

Φ'-valued Gaussian random measure Let $T = \mathbb{R}_+$ and $A = B(\mathbb{R}_+)$. A Φ'-valued set function $W(\cdot)$ on (T, A) is said to be a Φ'-valued Gaussian random measure if for each $\phi \in \Phi$, $W(\cdot)[\phi]$ is a real valued Gaussian random measure on (T, A). For $r \in \mathbb{R}$ we denote by $L^2(\Omega \to H_r)$ the Hilbert space (identifying

elements which are equal a.e. dP) of H_r-valued random elements G such that $E||G||_r^2 < \infty$.

<u>Proposition 4.1.4</u> Let (W_t) $t \geq 0$ be a Φ'-valued Wiener process as in Assumption 4.1.2. Define for $A = (s,t]$ $0 < s < t$

$$W(A) = W_t - W_s \quad .$$

Then $W(\cdot)$ can be extended to a Φ'-valued Gaussian random measure on $(T,\mathcal{A})$, which is an orthogonally scattered measure in $L^2(\Omega \to H_{-q})$, for $q \geq r_1 + r_2$, and control measure $\mu(\cdot) = \theta_q m(\cdot)$ where m denotes the Lebesgue measure on $(T,\mathcal{A})$ and

$$\theta_q = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} Q(\phi_j,\phi_j) < \infty \; .$$

Moreover, if $A \in \mathcal{A}$ $m(A) < \infty$ and $\phi \in H_q$, $W(A)[\phi] \in \mathcal{H}$, and if $B \in \mathcal{A}$ $m(B) < \infty$ then for all ϕ, $\psi \in H_q$

(4.1.27) $$E(W(A)[\phi]W(B)[\psi]) = m(A \cap B)\, Q\,(\phi,\psi) \, .$$

<u>Proof</u> From Theorem 4.1.1 $(W_t)_{t \in T}$ has an H_{-q}-valued continuous version for $q \geq r_1 + r_2$. Let $\{(1+\lambda_j)^q \phi_j\}_{j \geq 1}$ be a CONS for H_{-q}. Then if $A = (s,t]$, $W(A) = W_t - W_s$

$$E||W(A)||_{-q}^2 = E(\sum_{j=1}^{\infty} (1+\lambda_j)^{2q} \langle W(A),\phi_j\rangle_{-q}^2)$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} E(W(A)[\phi_j])^2 = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} m(A)\, Q\,(\phi_j,\phi_j) \, .$$

Let $\theta_q = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} Q(\phi_j,\phi_j)$ then

$$\theta_q \leq \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} \theta_2 ||\phi_j||_{r_2}^2 \leq \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(q-r_2)} \leq \theta_2 \theta_1 < \infty$$

and hence $E||W(A)||_{-q}^2 = \theta_q m(A) = \mu(A) \, .$

Thus $W(\cdot)$ can be extended to an orthogonally scattered measure on (T,A) with values in the Hilbert space $L^2(\Omega\to H_{-q})$ and control measure $\mu(\cdot) = \theta_q m(\cdot)$.

Next if $A \in A$ $m(A) < \infty$ and $\{A_n\}_{n\geq 1}$ is a sequence of disjoint sets in A with $m(A_n) < \infty$ all $n \geq 1$ and $A = \overset{\infty}{\underset{n=1}{\cup}} A_n$, then for all $\phi \in H_q$

$$E(W(A)[\phi] - \sum_{j=1}^{n} W(A_j)[\phi])^2 \leq ||\phi||_q^2 E||W(A) - \sum_{j=1}^{n} W(A_n)||_{-q}^2 \to 0 \quad \text{as } m\to\infty$$

i.e., $W(\cdot)[\phi]$ is a real valued Gaussian random measure with values in H. Let $\{\tilde{\phi}_j = (1+\lambda_j)^{-q}\phi_j\}$ be a CONS for H_q. Taking $A \in A$, $m(A) < \infty$, $W(A)[\phi_j]_{j\geq 1}$ and using a similar argument to the one used in the proof of Theorem 4.1.1 one shows that W is a Φ'-valued Gaussian random measure. Finally, (4.1.27) follows by applying the approximation theorem to A, $B \in A$, $m(A) < \infty$, $m(B) < \infty$ and using Corollary 4.1.2.

Q.E.D.

Now let $T = [0,T_o]$, $T_o > 0$ and $A = B(T)$. For $q \geq r_1 + r_2$ let $\{e_k\}_{k\geq 1}$ be a CONS for H_q. Then from the above proposition we are able to obtain an infinite system of independently scattered Gaussian random measures $\{W(\cdot)[e_k]\}_{k\geq 1}$ on (T,A) with values in H, that are mutually independent over disjoint sets, each one with control measure $\mu_k(\cdot) = m(\cdot)\,Q\,(e_k,e_k)$ and such that for all $k,j \geq 1$ and A, $B \in A$.

(4.1.28) $\quad E(W(A)[e_k]W(B)[e_j]) = m(A\cap B)\,Q\,(e_j,e_k)$.

Then for each set of index $k_1,\ldots,k_n$ in $\{1,2,\ldots\}$ we may construct the symmetric tensor product measure of $W(\cdot)[e_{k_1}],\ldots,W(\cdot)[e_{k_n}]$, denoted by $\overset{n}{\underset{i=1}{\circledast}} W[e_{k_i}]$, as in Sections 2.2 and 3.1 of this work, and construct multiple integrals with respect to it. In the next result we summarize the main properties of $\overset{n}{\underset{i=1}{\circledast}} W[e_{k_i}]$ and their integrals. It will be used in

Propositions 5.1.3 and 5.1.4.

Lemma 4.1.5 Let $T=[0,T_o]$, $T_o>0$, $\mathcal{A}=\mathcal{B}(T)$ and H be the Gaussian space defined in (4.1.25). For $q\geq r_1+r_2$ let $\{e_k\}_{k\geq 1}$ be a CONS in H_q. For a finite collection of index $k_1,\ldots,k_n$ in $\{1,2,\ldots\}$ denote by $\overset{n}{\underset{i=1}{\circledcirc}} W[e_{k_i}]$ the symmetric tensor product measure of $W(\cdot)[e_{k_1}],\ldots,W(\cdot)[e_{k_n}]$ given by Theorem 2.2.1. Then

a) $\overset{n}{\underset{i=1}{\circledcirc}} W[e_{k_i}]$ is an $H^{\circledcirc n}$-valued measure on $(T^n,\mathcal{A}^n)$.

b) $E(\overset{n}{\underset{i=1}{\circledcirc}} W[e_{k_i}](A))=0$ for each $A\in\mathcal{A}^n$.

c) A function $f\colon T^n\to\mathbb{R}$ is $\overset{n}{\underset{i=1}{\circledcirc}} W[e_{k_i}]$-integrable (see Definition 2.3.1) if and only if $f\in L^2(T^n,\mathcal{A}^n,m^{\otimes n})$. Denote by $\int_{T^n} f(\underline{t})d\overset{n}{\underset{i=1}{\circledcirc}} W[e_{k_i}](\underline{t})$ this integral.

d) Let $f\in L^2(T^n,\mathcal{A}^n,m^{\otimes n})$ and $g\in L^2(T^\ell,\mathcal{A}^\ell,m^{\otimes \ell})$.

Then for each collection of index $j_1,\ldots,j_n$, $k_1,\ldots,k_\ell$ in $\{1,2,\ldots\}$

$$E\Big(\int_{T^n} f(\underline{t})d\overset{n}{\underset{i=1}{\circledcirc}} W[e_{j_i}](\underline{t}) \int_{T^\ell} g(\underline{t})d\overset{\ell}{\underset{i=1}{\circledcirc}} W[e_{k_i}](\underline{t})\Big)$$

$$=\delta_{n\ell}\, n!\, Q^{\otimes n}(e_{j_1}\otimes\cdots\otimes e_{j_n}, e_{k_1}\otimes\cdots\otimes e_{k_n}) \int_{T^n}\tilde{f}(\underline{t})\tilde{g}(\underline{t})d\underline{t}$$

where $\tilde{f}(\underline{t}) = \frac{1}{n!}\sum_{\Pi} f(\underline{t}_\Pi)$.

Proof Since for each $k\geq 1$ $W(\cdot)[e_k]$ is a zero mean H-valued independently scattered measure then (a) follows by using Theorem 2.2.1. The proof of (b) follows since $H^{\circledcirc n}$ is orthogonal to $\overline{sp}\{1\}$ in $L^2(\Omega,\mathcal{F}^W,P)$ and by using (a). Next since $\mu_k(\cdot)=m(\cdot)\,Q\,(e_k,e_k)$ where m is the Lebesgue measure on $(T,\mathcal{A})$, the proof of (c) follows by Theorem 2.3.2.

Finally, the proof of (d) is similar to the proof of Theorem 2.3.3,

first using A^n-simple functions and then using approximation arguments.

Q.E.D.

If $T_o = 1$ and $Q(e_k, e_k) = 1$, all $k \geq 1$, $\{W(\cdot)[e_k]\}_{k\geq 1}$ is a sequence of orthogonally scattered measures in H of the kind considered in Theorem 2.1.4. Then we can construct the infinite tensor product measure $\overset{\infty}{\underset{k=1}{\otimes}} W[e_k]$ on $(T^\infty, A^\infty, m^\infty)$ with values in $\overset{\infty \underline{u}}{\underset{i=1}{\otimes}} H$, $\underline{u} = (W(T)[e_k]\ k \geq 1)$ (see Theorem 2.1.4). However, the construction of the infinite symmetric tensor product measure is a problem that remains open.

4.1.3 Examples

We now consider some examples of Φ'-valued Wiener processes.

Example 4.1.4 (Kallianpur and Wolpert (1984)). Let Φ be defined in Example 4.1.1 where $H_o = L^2(X, d\Gamma)$ for Γ a σ-finite measure on (X, Σ). Let μ be a σ-finite measure on $\mathbb{R} \times X$ such that the bilinear form

$$Q(\phi, \psi) = \int_{\mathbb{R} \times X} a^2 \phi(x) \psi(x) \mu(da, dx)$$

on $\Phi \times \Phi$ is continuous. In connection with neurophysiology applications Kallianpur and Wolpert (1984) give several examples of Q in which the measure μ is of the form

$$\mu(A \times B) = \sum_{k=1}^{p1} 1_A(a_e^k) \nu_e^k(B) + \sum_{\ell=1}^{p2} 1_A(-a_i^\ell) \nu_i^\ell(B) \quad A \in \mathcal{B}(\mathbb{R}),\ B \in \Sigma$$

where $\{a_e^k, a_i^k\}$ are positive real numbers and $\{\nu_e^k, \nu_i^\ell\}$ are finite measures on (X, Σ). The authors consider a W. path-continuous Φ'-valued independent increments process with characteristic functional

$$E(e^{iW_t[\phi]}) = e^{itm[\phi] - \frac{1}{2} t\, Q(\phi, \phi)} \qquad \phi \in \Phi, \quad t \geq 0$$

where $m \in \Phi'$. Theorem 4.1.1 enables us to construct such a process with continuous paths lying in H_{-q} for any $q \geq r_1 + r_2$ if m and Q satisfy

$$|m[\phi]|^2 + Q(\phi,\phi) \leq \theta_2 ||\phi||^2_{r_2} \qquad \phi \in \Phi$$

for some positive constants θ_2 and r_2.

The continuous positive definite bilinear form Q on $\Phi \times \Phi$ is not necessarily diagonalized by the system $\{\phi_j\}_{j\geq 1}$ of eigenvectors of $-L$. If for example

$$\Gamma(dx) = \int_{\mathbb{R}} a^2 \mu(da,dx)$$

then $Q(\phi_i,\phi_j) = \int a^2 \phi_i(x)\phi_j(x)\mu(dadx) = 0$ for $i \neq j$; but in general this is not the case.

<u>Example 4.1.5</u> (Itô (1978a)). Let Φ be as in Example 4.1.2, i.e. $\Phi = S(\mathbb{R}^d)$ $d \geq 1$. Itô (1978a) gives the following examples:

i) The Wiener Φ'-valued process corresponding to $Q_1(\phi,\psi) = \langle\phi,\psi\rangle_o$ is called a standard Φ'-valued process and it is denoted by $(b_t)_{t\in\mathbb{R}_+}$. By Corollary 4.1.1 (b) is an H_{-1}-valued Wiener process, since for the Example 4.1.2 $r_1 = 1$.

(ii) Let $\Delta = \sum_{i=1}^{d} \partial_i^2$ and define $\Delta b_t = \Delta b_t[\phi] \equiv b_t(\Delta\phi)$. Then $W_t \equiv \Delta b_t$ is an H_{-2}-valued Wiener process for which

$$Q_2(\phi,\psi) = \langle\Delta\phi,\Delta\psi\rangle_o \qquad \phi,\ \psi \in \Phi .$$

(iii) Let $1b_t = (b_t^1, b_t^2, \ldots, b_t^d)$ where the component processes b_t^i $i=1,\ldots,d$ are independent standard Φ'-valued Wiener processes. Then

$$W_t = \sum_{i=1}^{d} \partial_i b_t^i$$

is an H_{-3}-valued Wiener process with

$$Q_3(\phi,\phi) = -\langle\Delta\phi,\phi\rangle_0 \qquad \phi \in \Phi$$

where Δ is defined in (ii).

We observe that while Q_1 is diagonalized by $\{\phi_j\}_{j\geq 1}$, Q_2 and Q_3 are not.

Example 4.1.6 Φ'-valued Wiener processes arise in a natural way in the following manner. Let A be a relatively compact subset of $\mathbb{R}^d$ $d \geq 1$. Let $\tilde{W}(t,\underline{x})$ $t \in T$, $\underline{x} \in A$ be a two parameter sample continuous centered Gaussian system of random variables such that

$$E(\tilde{W}(t,\underline{x})\tilde{W}(s,\underline{y})) = \min(s,t)V(\underline{x},\underline{y}) \qquad t,\ s \in \mathbb{R}_+ \quad \underline{x},\ \underline{y} \in A$$

where V is square integrable over $A \times A$. Define for s, $t \in \mathbb{R}_+$

$$W_t[\phi] = \int_A \tilde{W}(t,\underline{x})\phi(\underline{x})d\underline{x}$$

for ϕ in a suitable class, $\Phi = S(\mathbb{R}^d)$ for example. Then from Theorem 4.1.1 we have that $\{W_t\}_{t\in\mathbb{R}_+}$ is a Φ'-valued Wiener process such that for s, $t \in \mathbb{R}_+$ and ϕ, $\psi \in \Phi$

$$E(W_t[\phi]W_s[\psi]) = \min(s,t)\,Q(\phi,\psi)$$

where

$$Q(\phi,\psi) = \int\int_{A\times A} V(\underline{x},\underline{y})\phi(\underline{x})\phi(\underline{y})d\underline{x}d\underline{y}.$$

is not necessarily diagonalized by the system $\{\phi_j\}_{j\geq 1}$, which in the case of $\Phi = S(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} \phi_i(\underline{x})\phi_j(\underline{x})d\underline{x} = 0 \qquad \text{for } i \neq j.$$

Example 4.1.7 (Cylindrical Brownian motion) Let K be a Hilbert space, (Ω,F,P) a complete probability space and $(F_t)_{t\geq 0}$ be an increasing family

of sub σ-fields of F. A measurable mapping

$$B_t(k,w) = [0,\infty) \times K \times \Omega \to \mathbb{R}$$

is called an F_t-cylindrical Brownian motion on K (Yor (1974)) (c.B.m.) if it satisfies the following two conditions:

a) For each $k \in K$, $k \neq 0$, $B_t(k)/\|k\|_K$ is a one dimensional F_t-Brownian motion.

b) $B_t(k)$ is linear in $k \in K$.

Two well-known observations (Miyahara (1981)) are the following: 1) A c.B.m. (B_t) cannot be regarded as a process on K, i.e. it is not a K-valued process; 2) if k_1 and k_2 are orthogonal elements in K, then $\{B.(k_1)\}$ and $\{B.(k_2)\}$ are independent.

Using the notation of Example 4.1.1 we now study the following case considered by Miyahara (1981). Let $H_o = L^2([0,\pi])$, $L = \hat{\omega}$, $\hat{\omega} = \sqrt{-\Delta}$ where Δ is the Laplacian ($\Delta = d/dx^2$) on H_o. Hence for j=1,2,... $\lambda_j = j$ and

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \cos(jx).$$

Then construct Φ as in Example 4.1.1. Miyahara (1981) considers a cylindrical Brownian motion B_t on H_o. Then $\{B_t(\phi): \phi \in \Phi\ t \geq 0\}$ is a centered Gaussian system of random variables such that for ϕ, $\psi \in \Phi$ and s, $t \in \mathbb{R}_+$

$$E B_t(\phi) B_s(\psi) = \min(s,t)\, Q(\phi,\psi)$$

where $Q(\phi,\psi) = \langle\phi,\psi\rangle_o$ ϕ, $\psi \in \Phi$. Then $r_2 = 0$ and since

$$\sum_{j=1}^{\infty} (1+j)^{-2r} < \infty$$

for $r > \frac{1}{2}$ then $r_1 = \frac{1}{2}$. Thus, applying Theorem 4.1.1 there exists a

Φ'-valued Wiener process W_t such that

$$W_t[\phi] = B_t(\phi) \qquad \text{a.s.}$$

for $t \geq 0$ and $\phi \in \Phi$. Moreover, using Corollary 4.1.1 W_t has an $H_{-\frac{1}{2}}$-valued continuous version.

Note that in this particular example the continuous positive definite bilinear form Q on $\Phi \times \Phi$ is diagonalized by the eigenvector system $\{\phi_j\}_{j \geq 1}$.

Example 4.1.8 (Independent system of one dimensional Brownian motions (Hitsuda and Watanabe (1978))). Let Φ be the countably Hilbert nuclear space defined in Example 4.1.3, i.e. $\Phi = S(\mathbb{Z})$. Let $\{B_t^{(i)}\}$ $t \geq 0$ $i \geq 1$ be a system of independent one dimensional Brownian motions on a complete probability space (Ω, F, P). For $\phi \in \Phi$, $\phi = (\phi^1, \phi^2, \ldots)$ define

$$W_t[\phi] = \sum_{i=1}^{\infty} B_t^{(i)} \phi^i .$$

Then since

$$E\left(\sum_{i=1}^{\infty} B_t^{(i)} \phi^i\right)^2 = t \sum_{i=1}^{\infty} (\phi^i)^2 < \infty \qquad t \geq 0$$

we obtain that since $\langle \phi, \psi \rangle_o = \sum_{i=1}^{\infty} \phi^i \psi^i$

$$E(W_t[\phi] W_s[\psi]) = \min(s,t) \langle \phi, \psi \rangle_o .$$

Hence, using Theorem 4.1.1 W_t has a version which is a $S(\mathbb{Z})$-valued Wiener process with c.p.d.b. form $Q(\cdot,\cdot) = \langle \cdot, \cdot \rangle_o$. Moreover, using Corollary 4.1.1 W_t has an H_{-1}-valued continuous version.

Example 4.1.9 (Finite dimensional case). Suppose Φ is finite dimensional, then $\Phi = H_o = \Phi' = \mathbb{R}^m$ say. Then $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ $t \geq 0$, where each $W_t^{(i)}$ is a Gaussian process with independent increments and

$$E(W_t^{(i)} W_s^{(j)}) = r_{ij} \min(s,t)$$

where $R = (r_{ij})$ is an $m \times m$ positive definite matrix. Then if $\phi = \sum_{j=1}^{m} \phi^j e_j$, $e_j = (0,\ldots,\underset{j}{1},\ldots,0)$ $\phi_j \in \mathbb{R}$ $j=1,\ldots,m$

$$Q(\phi,\psi) = \sum_{i=1}^{m} \sum_{j=1}^{m} \phi^i \psi^j r_{ij} .$$

Stochastic processes of this type have been considered in Chapter III.

4.2 Stochastic integrals

Throughout this section we will assume that (W_t) $t \geq 0$ is a Φ'-valued Wiener process with a c.p.d.b. form Q on $\Phi \times \Phi$, defined on a complete probability space (Ω, F, P), and that for each $t \geq 0$ $F_t = F_t^W = \sigma(W_s: 0 \leq s \leq t)$, with F_o containing all P-null sets of F. Also we make Assumptions 4.1.1 and 4.1.2 of Section 4.1. We recall that from Theorem 4.1.1 (W_t) $t \geq 0$ has an H_{-q} continuous version for $q \geq r_1 + r_2$.

Stochastic integrals with respect to $S(\mathbb{R}^d)'$-valued Wiener processes and E'-valued (E is a CHNS) processes have been discussed in Itô (1978a) and Mitoma (1981b) respectively. They propose to use the theory of stochastic integration on Hilbert spaces, as presented for example in Kunita (1970) or Kuo (1975), to construct stochastic integrals for the H_{-q} valued Wiener process (W_t) $t \geq 0$. Here we construct "weak" stochastic integrals similar to the case of a cylindrical Brownian motion as presented in Yor (1974). However, we do not work with a cylindrical Brownian motion but rather with an H_{-q}-valued Wiener process. Secondly, if $\{e_k\}_{k \geq 1}$ is any CONS in H_q, then $\{W_t[e_k]\}_{k \geq 1}$ is not necessarily a system of independent random variables (see Corollary 4.1.2), as it would be required in the case of a cylindrical Brownian motion (see Example 4.1.7). Moreover, we do not assume that the common orthogonal system in H_r $r \geq 0$ $\{\phi_j\}_{j \geq 1}$ (the

eigenvectors of the infinitesimal generator L) diagonalizes the c.p.d.b. form Q. The case when $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$, and then $\{\phi_j\}_{j\geq 1}$ diagonalizes Q, has been considered by Daletskii (1967) and Miyahara (1981) (see Example 4.1.7). Nevertherless, the nuclearity of the space $(\Phi, ||\cdot||_r \; r \geq 0)$ enables us to construct "weak" integrals even when $\{W_t[\phi_j]\}_{j\geq 1}$ is not an independent system of random variables.

We present real valued (Section 4.2.1) and Φ'-valued (Section 4.2.2) stochastic integrals with respect to W_t. They will be useful in Chapter V in studying real valued and Φ'-valued nonlinear functionals of W. We make extensive use of the c.p.d.b. form Q on $\Phi \times \Phi$ and its associated Rigged Hilbert Space (see (4.1.20)). This is motivated from Definition 4.1.2 which suggests that $L^2(\mathbb{R}_+) \otimes H_Q$ should play the role of the Reproducing Kernel Hilbert space, a concept that has been useful in studying nonlinear functionals of Gaussian processes (see Chapter VI of Kallianpur (1980)).

4.2.1 Stochastic integrals for Φ-valued random integrands (Real valued stochastic integrals)

The aim of this section is to define stochastic integrals for Φ-valued non-anticipative functions.

Definition 4.2.1 Let K be a real separable Hilbert space. A function $f\colon [0,\infty) \times \Omega \to K$ is said to belong to the class $M(W,K)$ if f is an F_t-adapted measurable (non-anticipative) function on $\mathbb{R}_+ \times \Omega$ to K such that for each $t > 0$

$$(4.2.1) \qquad \int_0^t E||f(s)||_K^2 \, ds < \infty .$$

The special classes we will be concerned with are $M_q = M(W,H_q)$, $q \geq r_1 + r_2$ and $M_Q = M(W,H_Q)$.

Stochastic integrals for elements in $\mathcal{M}_q$ $q \geq r_1 + r_2$

Definition 4.2.2 Let $q \geq r_1 + r_2$. For $g \in \mathcal{M}_q$ and $t > 0$ define the stochastic integral $\int_0^t \langle g_s, dW_s\rangle_q$ as

$$(4.2.2) \qquad \int_0^t \langle g_s, dW_s\rangle_q = \sum_{i=1}^{\infty} \int_0^t \langle g_s, e_i\rangle_q dW_s[e_i]$$

where $\{e_i\}_{i\geq 1}$ is a CONS for H_q and the integrals on the right hand side of (4.2.2) are Itô integrals.

Proposition 4.2.1 Let $g \in \mathcal{M}_q$ $q \geq r_1 + r_2$. Then the integral (4.2.2) is a well defined element in $L^2(\Omega, F^W, P)$. If $q_1 \geq r_1 + r_2$ and $g \in \mathcal{M}_{q_1}$ then this integral is independent of q and q_1. Moreover the following properties are satisfied for $f, g \in \mathcal{M}_q$.

a) For $a, b \in \mathbb{R}$ and $t > 0$

$$\int_0^t \langle af_s + bg_s, dW_s\rangle_q = a \int_0^t \langle f_s, dW_s\rangle_q + b \int_0^t \langle g_s, dW_s\rangle_q \quad \text{a.s.}.$$

b) $$E\left(\int_0^t \langle g_s, dW_s\rangle_q\right) = 0 \quad t > 0.$$

c) $$E\left(\int_0^{t_1} \langle g_s, dW_s\rangle_q \int_0^{t_2} \langle f_s, dW_s\rangle_q\right) = E \int_0^{t_1 \wedge t_2} Q(f_s, g_s) ds \quad t_1 > 0,\ t_2 > 0.$$

d) $$E\left(\int_0^t \langle f_s, dW_s\rangle_q\right)^2 = E \int_0^t Q(f_s, f_s) ds \leq E \int_0^t \| f_s \|_q^2 \, ds < \infty .$$

Proof We first prove that for $t > 0$ $\int_0^t \langle g_s, dW_s\rangle_q$ is a well defined element in $L^2(\Omega, F^W, P)$. Let $\{e_i\}_{i\geq 1}$ be any CONS for H_q $q \geq r_1 + r_2$. Then for each $t > 0$

$$g(t,\omega) = \sum_{j=1}^{\infty} \langle g_t(\omega), e_j\rangle_q \, e_j$$

and for $n, m \geq 1$, using Lemma 4.1.2

$$E\Big(\sum_{j=m}^{n}\int_{o}^{t}\langle g_s,e_j\rangle_q\, dW_s[e_j]\Big)^2 =$$

$$\sum_{j=m}^{n}\sum_{k=m}^{n} E\Big(\int_{o}^{t}\langle g_s,e_j\rangle_q\, dW_s[e_j]\int_{o}^{t}\langle g_s,e_k\rangle_q\, dW_s[e_k]\Big) =$$

$$\sum_{j=m}^{n}\sum_{k=m}^{n} E\int_{o}^{t}\langle g_s,e_j\rangle_q\langle g_s,e_k\rangle_q\, Q(e_j,e_k)ds.$$

Then since Q is a bilinear form, using (4.1.21) we obtain

(4.2.3)
$$E\Big(\sum_{j=m}^{n}\int_{o}^{t}\langle g_s,e_j\rangle_q\, dW_s[e_j]\Big)^2$$

$$= E\int_{o}^{t} Q\Big(\sum_{j=m}^{n}\langle g_s,e_j\rangle_q\, e_j,\sum_{j=m}^{n}\langle g_s,e_j\rangle_q\, e_j\Big)ds$$

$$\le \theta_2\, E\int_{o}^{t}\Big\|\sum_{j=m}^{n}\langle g_s,e_j\rangle_q\, e_j\Big\|_q^2\, ds$$

$$= \theta_2\, E\int_{o}^{t}\Big(\sum_{j=m}^{n}\langle g_s,e_j\rangle_q^2\Big)ds \qquad (e_j\text{'s are orthonormal})$$

$$\to 0 \quad \text{as } n,m\to\infty \quad \text{since } g\in M_q.$$

Thus $\int_o^t \langle g_s,dW_s\rangle_q$ is an element of $L^2(\Omega,F^W,P)$ defined as the $L^2(\Omega)$-limit of the Cauchy sequence $\{\sum_{j=1}^{n}\int_o^t\langle g_s,e_j\rangle_q dW_s[e_j]\}_{n\ge1}$.

The next argument will also show that (4.2.2) is independent of the CONS $\{e_j\}_{j\ge1}$ in H_q. Let $q_1\ge r_1+r_2$ $q_1\ge q$ and $\{\psi_j\}_{j\ge1}$ be a CONS for H_{q_1}. Then $\|\cdot\|_{r_2}\le\|\cdot\|_q\le\|\cdot\|_{q_1}$, $H_{q_1}\subset H_q\subset H_{r_2}$ and by Theorem 4.1.1, W_t has an H_{-q_1}-valued continuous version. Hence using Lemma 4.1.2, if $g\in M_q\cap M_{q_1}$

$$E\Big(\sum_{j=1}^{n}\int_{o}^{t}\langle g_s,e_j\rangle_q\, dW_s[e_j]-\sum_{j=1}^{n}\int_{o}^{t}\langle g_s,\psi_j\rangle_q\, dW_s[\psi_j]\Big)^2 =$$

$$E\Big(\int_{o}^{t}Q\Big(\sum_{j=1}^{n}\langle g_s,e_j\rangle_q\, e_j-\sum_{j=1}^{n}\langle g_s,\psi_j\rangle_{q_1}\psi_j,\sum_{j=1}^{n}\langle g_s,e_j\rangle_q\, e_j-\sum_{j=1}^{n}\langle g_s,\psi_j\rangle_{q_1}\psi_j\Big)\, ds\Big)$$

$$\le \theta_2 E\int_{o}^{t}\Big\|\sum_{j=1}^{n}\langle g_s,\psi_j\rangle_{q_1}\psi_j-\sum_{j=1}^{n}\langle g_s,e_j\rangle_q\, e_j\Big\|_q^2\, ds$$

$\to 0$ as $n\to\infty$ by dominated convergence theorem since

$$\|\sum_{j=1}^{n} \langle g_s,\psi_j\rangle_{q_1}\psi_j - \sum_{j=1}^{n} \langle g_s,e_j\rangle_q\|_q^2$$

$$\le 2(\|\sum_{j=1}^{n}\langle g_s,\psi_j\rangle_{q_1}\psi_j\|_q^2 + \|\sum_{j=1}^{n}\langle g_s,e_j\rangle_q e_j\|_q^2)$$

$$\le 2(\|\sum_{j=1}^{n}\langle g_s,\psi_j\rangle_{q_1}\psi_j\|_{q_1}^2 + \|\sum_{j=1}^{n}\langle g_s,e_j\rangle_q e_j\|_q^2)$$

$$\le 2(\|g_s\|_{q_1}^2 + \|g_s\|_q^2) \quad \text{all } n\ge 1.$$

Hence the integral (4.2.2) is independent of q and q_1.

The proof of (a) follows by the linearity property of the ordinary Itô integral and the proof of (b) follows since for each $i\ge 1$ $E(\int_o^t\langle f_s,e_i\rangle_q dW_s[e_i]) = 0$. The proof of (c) follows by using Lemma 4.1.2:

$$E(\int_o^{t_1}\langle g_s,dW_s\rangle_q \int_o^{t_2}\langle f_s,dWs\rangle_q) =$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E(\int_o^{t_1}\langle g_s,e_j\rangle_q dW_s[e_j] \int_o^{t_2}\langle f_s,e_k\rangle_q dW_s[e_k]) =$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E\int_o^{t_1\wedge t_2}\langle g_s,e_j\rangle_q\langle g_s,e_k\rangle_q Q(e_j,e_k)ds =$$

$$E\int_o^{t_1\wedge t_2} Q(f_s,g_s)ds \qquad t_1,t_2\ge 0.$$

Finally (d) is obtained by (c) and (4.1.21).

Q.E.D.

<u>Proposition 4.2.2</u> Let $f\in \mathcal{M}_q$ for $q\ge r_1+r_2$. Then the real valued process $\{\int_o^t\langle f_s,dW_s\rangle_q\}_{t\ge 0}$ is an F_t-martingale with associated increasing process

(4.2.4) $$E\int_o^t Q(f_s,f_s)ds .$$

For $t\in T=[0,T_o]$, $T_o>0$, it is a square integrable martingale with

a continuous modification.

Proof For $t \geq 0$ and $\{e_i\}_{i\geq 1}$ a CONS for H_q $q \geq r_1 + r_2$ let

$$Y_t = \sum_{j=1}^{\infty} \int_0^t \langle f_s, e_j \rangle_q \, dW_s[e_j]$$

and for $n \geq 1$ write $Y_t^n = \sum_{j=1}^n \int_0^t \langle f_s, e_j \rangle_q \, dW_s[e_j]$.

In the proof of Proposition 4.2.1 we have shown that for each $t > 0$ $Y_t^n \to Y_t$ in mean square. Next since for each $i \geq 1$ $\int_0^t \langle f_s, e_i \rangle_q \, dW_s[e_i]$ is an Itô integral, then for each $n \geq 1$

$$E(Y_t^n | F_s) = Y_s^n \quad \text{a.s.} \quad s < t$$

and

$$E(Y_s^n - E(Y_t | F_s))^2 = E(E(Y_t^n | F_s) - E(Y_t | F_s))^2$$

$$= E(E(Y_t^n - Y_t) | F_s)^2 \leq E(E(Y_t^n - Y_t)^2 | F_s) = E(Y_t - Y_t^n)^2 \underset{n\to\infty}{\to} 0.$$

Thus for $s < t$ $E(Y_t | F_s) = Y_s$ a.s. and from Proposition 4.2.1(c) for $t \in T = [0, T_o]$, $(Y_t)_{t\in T}$ is a square integrable martingale with increasing process $E\int_0^t Q(f_s, f_s) ds$.

Next since each Y_t^n has a continuous modification, then for each $n, m \geq 1$ $|Y_t^n - Y_t^m|$ is a continuous non-negative submartingale and by Doob's inequality

$$E[\sup_{t\in T} |Y_t^n - Y_t^m|^2] \leq 4E\, |Y_{T_o}^n - Y_{T_o}^m|^2 \to 0 \quad \text{as } n, m \to \infty .$$

Hence there exists a subsequence $\{Y_t^{n_k}\}$ that converges uniformly a.s. on T to a continuous version of Y_t.

Q.E.D.

Corollary 4.2.1 For $q \geq r_1 + r_2$ let $f: [0,\infty) \times \Omega \to H_q$ be a non-anticipative H_q-valued process such that

$$(4.2.5) \qquad \int_0^\infty E\|f(s)\|_q^2\,ds < \infty\,.$$

Define $\int_0^\infty \langle f_s, dW_s\rangle_q$ as the mean square limit of $\int_0^t \langle f_s, dW_s\rangle_q$ as $t \to \infty$. Then this integral is well defined and has the properties (a)-(d) of Proposition 4.2.1 writing ∞ instead of t. Moreover, for all $t > 0$

$$E\Big(\int_0^\infty \langle f_s, dW_s\rangle_q \,\Big|\, F_t\Big) = \int_0^t \langle f_s, dW_s\rangle_q \qquad \text{a.s.}$$

and $(\int_0^t \langle f_s, dW_s\rangle_q, F_t)_{t\ge 0}$ is a square integrable martingale with increasing process (4.2.4) and a continuous version on $\mathbb{R}_+$.

<u>Proof</u> First we observe that from (4.2.5) we obtain that $f \in M_q$ and therefore for each $t > 0$ the integral $Y_t = \int_0^t \langle f_s, dW_s\rangle_q$ is well defined. Next by (4.1.21) in Proposition 4.1.3 and (4.2.5)

$$\int_0^\infty E(Q(f_s, f_s))ds < \infty$$

and therefore from Proposition 4.2.4 $(Y_t, F_t)_{t\ge 0}$ is a square integrable martingale. Then $Y_\infty = \int_0^\infty \langle f_s, dW_s\rangle_q$ can be defined as the mean square limit of Y_t as $t \to \infty$ and it is such that $E(Y_\infty | F_t) = Y_t$ a.s. for each $t > 0$. The continuous version of Y_t is obtained as in the proof of Proposition 4.2.2 by writing Y_∞ instead of Y_{T_0}.

Q.E.D.

We will see in Corollary 5.1.5 of Chapter V that the stochastic integrals defined in the above corollary are dense in the space of real valued nonlinear functionals of $(W_t)_{t\ge 0}$.

<u>Stochastic integrals for elements in M_Q</u> For $f \in M_Q$ a stochastic integral of the form (4.2.1) cannot be defined since (W_t) $t \ge 0$ is not an H_Q-valued process. However, we are still able to define a stochastic integral for $f \in M_Q$ with the help of the following lemma.

<u>Lemma 4.2.1</u> Let $q \geq r_1 + r_2$ and $f \in \mathcal{M}_Q$. Then there exists a sequence $\{f_n\}_{n\geq 1}$ in $\mathcal{M}_q$ such that for each $t > 0$

$$\int_0^t E\| f(s)-f_n(s)\|_Q^2 \, ds \to 0 \quad \text{as } n\to\infty .$$

<u>Proof</u> Recall that $f \in \mathcal{M}_Q$ implies that f is non-anticipative and that for each $t > 0$

$$\int_0^t \| f(s)\|_Q^2 \, ds < \infty .$$

Next let $\{e_i\}_{i\geq 1}$ be a CONS for H_Q and let P_n be the orthogonal projector onto the span of $\{e_1,\ldots,e_n\}$. For each $t > 0$ by monotone convergence theorem

$$\int_0^t E\| f_s\|_Q^2 \, ds = \sum_{j=1}^{\infty} \int_0^t E(\langle f_s,e_j\rangle_Q^2)ds$$

and hence for each $t > 0$

$$\int_0^t E\| P_n f_s - f_s\|_Q^2 ds = \sum_{j=n+1}^{\infty} \int_0^t E(\langle f_s,e_j\rangle_Q^2)ds \to 0 \quad \text{as } n\to\infty .$$

Next for all $n \geq 1$ there exists a sequence $(\beta_k^n)_{k\geq 1}$ of non-anticipative step processes with values in the range of P_n (this is the finite dimensional case, see for example Lemma 4.3.2 in Strook and Varadhan (1979) or Lemma 1.1 in Ikeda and Watanabe (1981)) such that for each $t > 0$

$$\int_0^t E\| \beta_k^n(s) - P_n f_s\|_Q^2 \, ds < \frac{1}{k} \qquad k=1,2,\ldots .$$

Define the H_Q-valued step process

$$\alpha_n(t)(\omega) = \beta_n^n(t)(\omega) \qquad 0 \leq t < \infty \qquad \omega \in \Omega \qquad n \geq 1.$$

Then for all $t > 0$

$$\int_0^t E\| \alpha_n(s) - f_s\|_Q^2 \, ds \leq \int_0^t E\| \alpha_n(s) - P_n f_s\|_Q^2 \, ds$$

$$+\int_0^t E\|P_n f_s - f_s\|_Q^2\, ds \le \frac{1}{n} + \int_0^t E\|P_n f_s - f_s\|_Q^2\, ds \to 0 \quad \text{as } n\to\infty .$$

Thus we have shown that if $f \in M_Q$, for all $\varepsilon > 0$ there exists an H_Q-valued step process $\alpha(t,\omega)$ such that for each $t > 0$

$$\text{(4.2.6)} \qquad \int_0^t E\|\alpha(s)-f(s)\|_Q^2\, ds < \varepsilon/4$$

where

$$\alpha(s,\omega) = \alpha_{t_j}(\omega) \quad \text{a.s.} \quad t_j \le s < t_{j+1} \quad j=0,\dots,n-1$$

$$= \alpha_{t_n}(\omega) \quad \text{a.s.} \quad s \ge t_n$$

where $0 = t_o < t_1 < \dots < t_n < \infty$ and each α_{t_j} takes values in a finite dimensional subspace B_j of H_Q, it is F_{t_j}-measurable and $E\|\alpha_{t_j}\|_Q^2 < \infty$ for $j=1,\dots,n$.

Next for each $j=1,\dots,n$ let $\{e_1^j,\dots,e_{k_j}^j\}$ be an orthogonal basis for B_j. *Since H_q is dense in H_Q we can choose $\{\psi_1^j,\dots,\psi_{k_j}^j\}$ such that $\psi_\ell^j \in H_q$ and*

$$\|\psi_\ell^j - e_\ell^j\|_Q^2 < \frac{\varepsilon}{2k_j(t_{j+1}-t_j)E\|\alpha_{t_j}\|_Q^2} \quad \ell=1,\dots,k_j .$$

Each α_{t_j} can be written as

$$\alpha_{t_j}(\omega) = a_1^j(\omega)\, e_1^j + \dots + a_{k_j}^j(\omega)\, e_{k_j}^j$$

where

$$E\|\alpha_{t_j}\|_Q^2 = E((a_1^j)^2 + \dots + (a_{k_j}^j)^2) < \infty .$$

Define

$$\alpha_{t_j}^*(\omega) = a_1^j(\omega)\psi_1^j + \dots + a_{k_j}^j(\omega)\psi_{k_j}^j$$

then

$$E\|\alpha_{t_j}^*\|_q^2 \le E((a_1^j)^2 + \dots + (a_{k_j}^j)^2) \sum_{i=1}^{k_j} \|\psi_i^j\|_q^2 < \infty$$

and

$$E\|\alpha_{t_j} - \alpha_{t_j}^*\|_Q^2 = E\|\sum_{i=1}^{k_j} a_i^j(e_i^j - \psi_i^j)\|_Q^2$$

$$\leq E\Big(\sum_{i=1}^{k_j} |a_i^j| \, \| e_i^j - \psi_i^j \| \Big)^2$$

$$\leq \{E\Big(\sum_{i=1}^{k_j} (a_i^j)^2\Big)\}\{\sum_{i=1}^{k_j} \| e_i^j - \psi_i^j \|_Q^2\} < \frac{\varepsilon}{2(t_{j+1}-t_j)} .$$

Finally define

(4.2.7)
$$\alpha^*(s,\omega) = \begin{cases} \alpha^*_{t_j}(\omega) & t_j \leq s < t_{j+1} \quad j=1,\dots,n-1 \\ \\ \alpha^*_{t_n}(\omega) & s > t_n \end{cases}$$

which is an element of M_q. Then for each $f \in M_Q$ and $\varepsilon > 0$ there exists $\alpha^* \in M_q$ such that for each $t > 0$

$$\int_0^t E\| \alpha^*(t) - f(t) \|_Q^2 \, dt < \varepsilon$$

and the existence of the required sequence follows.

Q.E.D.

<u>Definition 4.2.3</u> Let $f \in M_Q$, then from Lemma 4.2.1 there exists a sequence of functions $\{f_n\}_{n\geq 1}$ in M_q for $q \geq r_1 + r_2$, such that for each $t > 0$

$$\int_0^t E\| f(s) - f_n(s) \|_Q^2 \, ds \to 0 \quad \text{as } n \to \infty .$$

Then by Proposition 4.2.1 for each $t > 0$

$$E\Big(\int_0^t \langle f_n(s) - f_m(s), dW_s \rangle_q \Big)^2 = \int_0^t E\| f_n(s) - f_m(s) \|_Q^2 \, ds \to 0 \quad n,m \to \infty .$$

Define for each $t > 0$ the stochastic integral $\int_0^t \langle f_s, dW_s \rangle_Q$ as the $L^2(\Omega)$-limit of the Cauchy sequence $\{ \int_0^t \langle f_n(s), dW_s \rangle_q \}_{n\geq 1}$. This integral satisfies (a)-(d) of Proposition 4.2.1 and Proposition 4.2.2 for elements in M_Q. If in addition f is such that

(4.2.8)
$$\int_0^\infty E\| f_s \|_Q^2 \, ds < \infty$$

then the stochastic integral $\int_0^\infty \langle f_s, dW_s \rangle_Q$ is defined (as in Corollary 4.2.1) as the mean square limit of $\int_0^t \langle f_s, dW_s \rangle_Q$ as $t \to \infty$.

The main properties of the above integral are summarized in the next corollary.

<u>Corollary 4.2.2</u> Let $f, g \in M_Q$. Then

a) If $a, b \in \mathbb{R}$ and $t > 0$

$$\int_0^t \langle af_s + bg_s, dW_s \rangle_Q = a \int_0^t \langle f_s, dW_s \rangle_Q + b \int_0^t \langle g_s, dW_s \rangle_Q \quad \text{a.s..}$$

b) $E\left(\int_0^t \langle f_s, dW_s \rangle_Q\right) = 0 \quad \text{all } t > 0.$

c) $E\left(\int_0^{t_1} \langle f_s, dW_s \rangle_Q \int_0^{t_2} \langle g_s, dW_s \rangle_Q\right) = E \int_0^{t_1 \wedge t_2} \langle f_s, g_s \rangle_Q ds \quad t_1, t_2 > 0.$

d) $E\left(\int_0^t \langle f_s, dW_s \rangle_Q\right)^2 = E \int_0^t \|f_s\|_Q^2 \, ds < \infty$.

e) If $E \int_0^\infty \|f_s\|_Q^2 \, ds < \infty$ then $\{\int_0^t \langle f_s, dW_s \rangle_Q, F_t\}_{t \geq 0}$

is a square integrable martingale with corresponding increasing process

$$\int_0^t E\|f_s\|_Q^2 \, ds$$

and a continuous modification on $\mathbb{R}_+$. Moreover, for $t > 0$

$$E\left(\int_0^\infty \langle f_s, dW_s \rangle_Q \middle| F_t\right) = \int_0^t \langle f_s, dW_s \rangle_Q \quad \text{a.s.}$$

and

$$E\left(\int_0^\infty \langle f_s, dW_s \rangle_Q\right)^2 = E \int_0^\infty \|f_s\|_Q^2 \, ds.$$

The proof follows by the above definition, Propositions 4.2.1 and 4.2.2 and Corollary 4.2.1.

4.2.2 Stochastic integrals for operator valued processes (Φ'-valued stochastic integrals)

Let $L(\Phi',\Phi')$ denote the class of continuous linear operators from Φ' to Φ'. In this section we study the stochastic integrals of $L(\Phi',\Phi')$-valued non-anticipative processes with respect to a Φ'-valued Wiener process with c.p.d.b. form Q on $\Phi\times\Phi$.

Definition 4.2.4 A function $f:[0,\infty)\times\Omega\to L(\Phi',\Phi')$ is said to belong to the class $O_Q(\Phi',\Phi')$ if f is an F_t-adapted measurable (non-anticipative) function on $[0,\infty)\times\Omega$ to $L(\Phi',\Phi')$ such that for each $t>0$

$$(4.2.9)\qquad E\int_0^t Q(f_s^*(\phi),f_s^*(\phi))\,ds<\infty \qquad \forall\ \phi\in\Phi$$

where $f_s^*:\Phi\to\Phi$ is the adjoint of f_s, i.e. f^* is defined by the relation $f(\psi)[\phi]=\psi[f^*(\phi)]$ $\quad\psi\in\Phi',\quad\phi\in\Phi.$

Lemma 4.2.2 Let $f\in O_Q(\Phi',\Phi')$. Then for each $t>0$ there exists $q_{t,f}\geq r_1+r_2$ such that

$$E\int_0^t \|f_s^*\|^2_{\sigma_2(H_{q_{t,f}},H_Q)}ds=E\int_0^t \|f_s\|^2_{\sigma_2(H_Q,H_{-q_{t,f}})}ds<\infty$$

where $\sigma_2(H_{q_{t,f}},H_Q)$ denotes the Hilbert space of Hilbert-Schmidt operators from $H_{q_{t,f}}$ to H_Q.

Proof For each $t>0$ and $\phi\in\Phi$ let

$$(4.2.10)\qquad V_t^2(\phi)=E\int_0^t Q(f_s^*(\phi),f_s^*(\phi))ds.$$

Then since $f\in O_Q(\Phi',\Phi')$ for each $t>0$

$$(4.2.11)\qquad V_t^2(\phi)<\infty\qquad\forall\ \phi\in\Phi.$$

We first show that for $t > 0$ $V_t(\phi)$ is a continuous function on Φ. Let $\phi_n \to \phi$ in Φ, then since $f^*_s \in L(\Phi,\Phi)$ and Q is Φ-continuous, using Fatou's lemma we have that

$$V_t(\phi) = \{ E \int_0^t \liminf Q(f^*_s(\phi_n), f^*_s(\phi_n))ds\}^{1/2}$$

$$\leq \{\liminf E \int_0^t Q(f^*_s(\phi_n), f^*_s(\phi_n))ds\}^{1/2} = \liminf V_t(\phi_n)$$

which shows that V_t is a lower semicontinuous function on Φ. Applying the triangle inequality we obtain that for $\phi, \psi \in \Phi$ $V_t(\phi+\psi) \leq V_t(\phi) + V_t(\psi)$ and clearly $V_t(a\phi) = |a|\ V_t(\phi)$ for $a \in \mathbb{R}$. Then by Lemma 4.1.1 $V_t(\phi)$ is a continuous function on Φ and there exist $V_{t,f} > 0$ and $\theta_{t,f} > 0$ such that

$$(4.2.12) \qquad V_t^2(\phi) \leq \theta^2_{t,f} \|\phi\|^2_{r_{t,f}} \qquad \forall\ \phi \in \Phi.$$

Next let $\{\phi_j\}_{j\geq1}$ and $\{\lambda_j\}_{j\geq1}$ be as in Assumption 4.1.1. Choose $q_{t,f} \geq r_{t,f} + r_1$ and write $\tilde{\phi}_j = (1+\lambda_j)^{-q_{t,f}}\phi_j$ $\quad j \geq 1$. Then $\{\tilde{\phi}_j\}_{j\geq1}$ is a CONS for $H_{q_{t,f}}$ and using (4.2.10) and (4.2.12) we have that

$$E \int_0^t (\sum_{j=1}^{\infty} Q(f^*_s(\tilde{\phi}_j), f^*_s(\tilde{\phi}_j)))ds = \sum_{j=1}^{\infty} E \int_0^t Q(f^*_s(\tilde{\phi}_j), f^*_s(\tilde{\phi}_j))ds$$

$$= \sum_{j=1}^{\infty} V_t^2(\tilde{\phi}_j) \leq \theta^2_{t,f} \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|^2_{r_{t,f}} = \theta^2_{t,f} \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(q_{t,f}-r_{t,f})}$$

$$\leq \theta^2_{t,f}\theta_1 < \infty .$$ Thus for each $t > 0$

$$(4.2.13) \qquad E \int_0^t \|f^*_s\|^2_{\sigma_2(H_{q_{t,f}}, H_Q)} ds = E \int_0^t (\sum_{j=1}^{\infty} Q(f^*_s(\tilde{\phi}_j), f^*_s(\tilde{\phi}_j)))ds < \infty .$$

Q.E.D.

<u>Proposition 4.2.3</u> Let $f \in \mathcal{O}_Q(\Phi',\Phi')$. Then for each $t > 0$ there exists a Φ'-valued element $Y_t(f)$ such that

$$(4.2.14) \qquad Y_t(f)[\phi] = \int_0^t \langle f_s^*(\phi), dW_s \rangle_Q \quad \text{a.s.} \ \forall\ \phi \in \Phi$$

where the RHS of (4.2.14) is the stochastic integral of Definition 4.2.3. Moreover, for each $T_o > 0$ there exists a positive integer $q_{T_o,f}$ such that $Y_t(f) \in H_{-q_{T_o,f}}$ a.s. for $0 \le t \le T_o$. $Y_t(f)$ is called the <u>Φ'-valued stochastic integral of f</u> w.r.t. W and is denoted by

$$Y_t(f) = \int_0^t f_s dW_s \ .$$

<u>Proof</u> We first note that for each $t > 0$ and $\phi \in \Phi$ $\int_0^t \langle f_s^*(\phi), dW_s \rangle_Q$ is defined in the sense of Definition 4.2.3 since $f_s^*(\phi)$ is non-anticipative and

$$\int_0^t Q(f_s^*(\phi), f_s^*(\phi))ds < \infty$$

i.e. $f_s^*(\phi) \in M_Q$.

Using the notation of the proof of Lemma 4.2.2, define

$$Y_t(f)[\tilde{\phi}_j] = \int_0^t \langle f_s^*(\tilde{\phi}_j), dW_s \rangle_Q \qquad j \ge 1.$$

Then by Corollary 4.2.2 (d), (4.2.10) and (4.2.12)

$$E\left(\sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2\right) = \sum_{j=1}^{\infty} E(Y_t(f)[\tilde{\phi}_j])^2 = \sum_{j=1}^{\infty} V_t^2(\tilde{\phi}_j)$$

$$\le \theta_{t,f}^2 \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|_{r_{t,f}}^2 \le \theta_{t,f}^2\ \theta_1 < \infty \ .$$

Thus $\sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2 < \infty$ a.s.. Let

$$\Omega_1 = \{\omega : \sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j](\omega))^2 < \infty\} \quad \text{then } P(\Omega_1) = 1.$$

Let $\{\psi_j\}_{j\ge1}$ be the CONS for $H_{-q_{t,f}}$ dual to $\{\tilde{\phi}_j\}_{j\ge1}$ and define

$$(4.2.15)\qquad \tilde{Y}_t(f)(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y_t[\tilde{\phi}_j](\omega)\psi_j & \omega\in\Omega_1 \\ 0 & \omega\notin\Omega_1 \end{cases}.$$

Then for each $t>0$ $\tilde{Y}_t(f)\in H_{-q_{t,f}}$ a.s. for $q_{t,f}\geq r_{t,f}+r_1$ and therefore $\tilde{Y}_t(f)\in\Phi'$ a.s. .

It remains to prove that $\tilde{Y}_t$ satisfies (4.2.14). Let $t>0$ and $\phi\in\Phi$, then $\phi\in H_{q_{t,f}}$ and

$$\phi = \lim_{n\to\infty}\sum_{j=1}^{n}\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}, \quad (\text{limit in } H_{q_{t,f}})$$

and therefore $\quad V_t(\sum_{j=1}^{n}\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}_j-\phi)\to 0$ as $n\to\infty$

which implies from (4.2.10) that

$$(4.2.16)\qquad E\int_0^t Q(f_s^*(\sum_{j=m}^{n}\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}_j),\ f_s^*(\sum_{j=m}^{n}\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}_j))ds\to 0 \quad \text{as } n,m\to\infty .$$

On the other hand, since $\psi_j[\phi]=\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}$ then

$$\tilde{Y}_t(f)[\phi]=\sum_{j=1}^{\infty}Y_t[\tilde{\phi}_j]\psi_j[\phi] = \sum_{j=1}^{\infty}Y_t[\tilde{\phi}_j]\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}$$

$$=\sum_{j=1}^{\infty}Y_t[\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}_j] \qquad \text{a.s. .}$$

Thus if $g_n(s) = f_s^*(\sum_{j=1}^{n}\langle\phi,\tilde{\phi}_j\rangle_{q_{t,f}}\tilde{\phi}_j)$

$$\tilde{Y}_t(f)[\phi] = \lim_{n\to\infty}\int_0^t\langle g_n(s),dW_s\rangle_Q \qquad \text{a.s.}$$

and from (4.2.16) and Definition 4.2.3

$$\int_0^t\langle g_n(s),dW_s\rangle_Q\to\int_0^t\langle f_s^*(\phi),dW_s\rangle_Q \quad \text{in } L^2(\Omega).$$

Thus for each $t>0$ $\tilde{Y}_t(f)[\phi] = \int_0^t\langle f_s^*(\phi),dW_s\rangle_Q$ a.s. $\forall\ \phi\in\Phi$. From now on

we write $Y_t(f)$ instead of $\tilde{Y}_t(f)$.

Q.E.D.

The Φ'-valued stochastic integral $Y_t(f) = \int_o^t f_s dW_s$ has the following properties.

<u>Proposition 4.2.4</u> Let $f,g \in \mathcal{O}_Q(\Phi',\Phi')$.

a) If $a,b \in \mathbb{R}$ then for each $t > 0$

$$Y_t(af+bg) = aY_t(f) + bY_t(g) \quad \text{a.s.} \; .$$

b) $E(Y_t(f)[\phi]) = 0 \quad \forall\ \phi \in \Phi \quad t > 0.$

c) $E(Y_t(f)[\phi]Y_t(f)[\psi]) = E\int_o^t Q(f_s^*(\phi), f_s^*(\psi))ds \quad \forall\ \phi,\psi \in \Phi.$

d) $E\|Y_t(f)\|^2_{-q_{t,f}} = E\int_o^t \|f_s\|^2_{\sigma_2(H_Q, H_{-q_{t,f}})} ds < \infty \quad \forall\ t > 0.$

e) $(Y_t(f), F_t)_{t\geq 0}$ is a Φ'-valued martingale. If $T = [0,T_o]$, $T_o > 0$ then $(Y_t(f), F_t)_{t\in T}$ is a Φ'-valued square integrable martingale with an $H_{-q_{T_o}}$ continuous version for some $q_{T_o} > 0$.

<u>Proof</u> (a), (b) and (c) follow from Proposition 4.2.3 and Corollary 4.2.2. To prove (d) let $\{\tilde{\phi}_j = (1+\lambda_j)^{q_{t,f}}\phi_j\}_{j\geq 1}$ be a CONS for $H_{-q_{t,f}}$. Then using monotone convergence theorem, (4.1.8), (4.2.14) and (c) above we have that

$$E\|Y_t(f)\|^2_{-q_{t,f}} = E(\sum_{j=1}^{\infty} \langle Y_t(f),\tilde{\phi}_j\rangle^2_{-q_{t,f}}) = \sum_{j=1}^{\infty} E\langle Y_t(f),(1+\lambda_j)^{q_{t,f}}\phi_j\rangle^2_{-q_{t,f}}$$

$$= \sum_{j=1}^{\infty}(1+\lambda_j)^{-2q_{t,f}} E(Y_t(f)[\phi_j])^2 = \sum_{j=1}^{\infty}(1+\lambda_1)^{-2q_{t,f}} E\int_o^t Q(f_s^*(\phi_j), f_s^*(\phi_j))ds$$

$$= E\int_o^t \sum_{j=1}^{\infty} Q(f_s^*(1+\lambda_j)^{-q_{t,f}}\phi_j, f_s^*(1+\lambda_j)^{-q_{t,f}}\phi_j))ds$$

$$= E\int_o^t \|f_s^*\|^2_{\sigma_2(H_{q_{t,f}},H_Q)} ds \quad (\{(1+\lambda_j)^{-q_{t,f}}\phi_j\}_{j\geq 1} \text{ is a CONS for } H_{q_{t,f}})$$

$$= E\int_0^t \|f_s\|^2_{\sigma_2(H_Q,H_{-q_{t,f}})}ds < \infty \qquad \text{(by Lemma 4.2.2).}$$

e) Let $V_t(\phi)$ be as in (4.2.10). Then for each $\phi \in \Phi$ $V_t(\phi)$ is a non-decreasing function of t. Hence from (4.2.12) there exist $\theta_{T_o,f} > 0$ and $r_{T_o,f} > 0$ such that for $\phi \in \Phi$

$$V_t^2(\phi) \le \theta^2_{T_o,f}\|\phi\|^2_{r_{T_o,f}} \qquad \forall\ t \in T = [0,T_o] \quad T_o > 0.$$

Then for each $t \in T$ $Y_t(f) \in H_{-q_{T_o,f}}$ a.s. for $q_{T_o,f} \ge r_{T_o,f} + r_1$ (see proof of Proposition 4.2.3). Then using (4.2.14), from Corollary 4.2.2 we have that for each $\phi \in \Phi$ $(Y_t(f)[\phi],F_t)_{t\ge 0}$ is a martingale, i.e. $(Y_t(f),F_t)_{t\ge0}$ is a Φ'-valued martingale. Moreover, it is a square integrable martingale in $T = [0,T_o]$ with associated increasing process

$$E\int_0^t Q(f_s^*(\phi),f_s^*(\phi))ds.$$

Next, using the notation as in the proof of Proposition 4.2.3

$$Y_t(f) = \sum_{j=1}^{\infty} Y_t(f)[\tilde{\phi}_j]\ \psi_j \qquad \text{a.s.}\ .$$

Therefore, from (4.2.14) and Corollary 4.2.2 for each $j \ge 1$ $Y_t(f)[\tilde{\phi}_j]$ has a continuous version on $T = [0,T_o]$ and therefore for each $n \ge 1$

$$M_n(t) = \sum_{j=1}^{n} Y_t[\tilde{\phi}_j]\psi_j$$

is an $H_{-q_{T_o,f}}$-valued martingale with a continuous version. Then using the usual argument, since $\|M_n(t)-M_m(t)\|_{-q_{T_o,f}}$ is a continuous non-negative submartingale, by Doob's inequality

$$E(\sup_{t\in T}\|M_n(t)-M_m(t)\|^2_{-q_{T_o,f}}) \le 4E\|M_n(T_o)-M_m(T_o)\|_{-q_{T_o,f}}$$

$$= 4E\|\sum_{j=n}^{m} Y_{T_o}(f)[\tilde{\phi}_j]\psi_j\|^2_{-q_{T_o},f} = 4E(\sup_{\|\phi\|_{q_{T_o},f}\le 1} |\sum_{j=n}^{m} Y_{T_o}(f)[\tilde{\phi}_j]\psi_j[\phi]|)$$

$$\le 4\sum_{j=n}^{m} E(Y_{T_o}(f)[\tilde{\phi}_j])^2 \le 4\sum_{j=n}^{m} V_{T_o}(\tilde{\phi}_j)$$

$$\le 4\,\theta^2_{T_o,f}\sum_{j=n}^{m}(1+\lambda_j)^{-2(q_{T_o,f}-r_{T_o,f})} \le 4\,\theta^2_{T_o,f}\sum_{j=n}^{m}(1+\lambda_j)^{-2r_1} \to 0$$

as $n,m \to \infty$.

Therefore, there exists a subsequence $M_{n_k}(t)$ that converges on $T=[0,T_o]$ uniformly to an $H_{-q_{T_o},f}$-continuous version of $Y_t(f)$.

Q.E.D.

We now extend the definition of $Y_t(\cdot)$ to functions which are integrable in $[0,\infty)\times\Omega$. Lemma 4.2.2 and Proposition 4.2.4 (d) suggest that it is enough to construct stochastic integrals for functions of the form $f:[0,\infty)\times\Omega \to \sigma_2(H_Q,H_{-r})$ for $r>0$ as we now do.

Let $r>0$. A function $f:[0,\infty)\times\Omega\to\sigma_2(H_q,H_{-r})$ is said to belong to the class $\mathcal{O}(H_Q,H_{-r})$ if f is an F_t-adapted measurable function on $\mathbb{R}\times\Omega$ to $\sigma_2(H_Q,H_{-r})$ such that

(4.2.17) $$\int_o^\infty E\|f_s\|^2_{\sigma_2(H_Q,H_{-r})}ds < \infty .$$

<u>Proposition 4.2.5</u> Let $r \ge r_1+r_2$ and $f\in\mathcal{O}(H_Q,H_{-r})$. Then there exists an H_{-r}-valued element $Y(f)$, called the <u>stochastic integral for elements in $\mathcal{O}(H_Q,H_{-r})$</u>, such that

(4.2.18) $$Y(f)[\phi] = \int_o^\infty \langle f^*_s(\phi),dW_s\rangle_Q \quad \text{a.s.} \quad \forall\, \phi\in H_r$$

where the RHS is the stochastic integral of Definition 4.2.3. We denote this integral by

$$Y(f) = \int_0^\infty f_s dW_s \ .$$

It has the following properties: If $f,g \in \mathcal{O}(H_Q,H_{-r})$

a) For $a,b \in \mathbb{R}$ $Y(af+bg) = aY(f)+bY(g)$ a.s..

b) $E(Y(f)[\phi]) = 0 \quad \forall\ \phi \in H_r$.

c) $E(Y(f)[\phi]Y(g)[\psi]) = E \int_0^\infty Q(f_s^*(\phi),g_s^*(\psi))ds \quad \phi,\ \psi \in H_r$.

d) $E\|Y(f)\|_{-r}^2 = E \int_0^\infty \|f_s\|_{\sigma_2(H_Q,H_{-r})}^2 ds < \infty$.

e) If $Y_t(f) = \int_0^t f(s)dW_s$, then $(Y_t(f),F_t^W)$ $t \geq 0$ is a Φ'-valued square integrable martingale with an H_{-r} continuous version.

<u>Proof</u> Taking $r_2 = r_{t,f}$ and $r = q_{t,f}$ all $t \geq 0$ as in the proof of Proposition 4.2.3 one shows that for each $t > 0$ the stochastic integral $Y_t(f) \in H_{-r}$ a.s. and it is such that

$$Y_t(f)[\phi] = \int_0^t \langle f_s^*(\phi),dW_s\rangle_Q \quad \text{a.s.} \quad \forall\ \phi \in H_r.$$

Then using Proposition 4.2.4 (d) and (4.2.17)

$$E\|Y_t - Y_{t'}\|_{-r}^2 = E \int_t^{t'} \|f_s\|_{\sigma_2(H_Q,H_{-r})}^2 ds \to 0 \quad \text{as } t' > t \to \infty \ .$$

Therefore there exists $Y(f) = Y_\infty(f)$ with the required property (4.2.18).

From (4.2.18) and Corollary 4.2.9 (a), (b) and (c) are proved. The proofs of (d) and (e) are similar to the proofs of (d) and (e) in Proposition 4.2.4 writing $r_2 = r_{T_o,f}$ and $r = q_{T_o,f}$.

Q.E.D.

CHAPTER V

MULTIPLE WIENER INTEGRALS FOR A NUCLEAR SPACE VALUED WIENER PROCESS

In this chapter we construct real valued (Section 5.1.1) and Φ'-valued (Section 5.2.1) multiple Wiener integrals with respect to a Φ'-valued Wiener process $(W_t)_{t\in\mathbb{R}_+}$ with a continuous positive definite bilinear form Q on $\Phi\times\Phi$, where Φ is the countably Hilbert Nuclear space of Section 4.1.1. We consider multiple Wiener integral expansions and stochastic integral representations for real valued (Section 5.1.2) and Φ'-valued (Section 5.2.2) nonlinear functionals of W. The Wiener decomposition of the space of Φ'-valued nonlinear functionals is obtained (Theorem 5.2.2) as well as representation theorems for real valued (Theorem 5.1.2) and Φ'-valued (Theorem 5.2.4) square integrable martingales.

Throughout the chapter we will assume that (Ω,F,P) is a complete probability space on which there is defined a Φ'-valued Wiener process $(W_t)_{t\in\mathbb{R}_+}$ with a c.p.d.b. form Q on $\Phi\times\Phi$, and for $t\geq 0$, $F_t = F_t^W = \sigma(W_s: 0\leq s\leq t)$ with F_o containing all P-null sets of F. Also we assume that θ_1, r_1, $\{\phi_j\}_{j\geq 1}$, $\{\lambda_j\}_{j\geq 1}$, H_r $-\infty< r<\infty$, and θ_2, r_2 are as in Assumptions 4.1.1 and 4.1.2.

5.1 Real valued multiple Wiener integrals

For $n\geq 1$ let $\Phi^{\otimes n}$ denote the n-fold tensor product of Φ (see Section 4.1.1). The aim of this section is to construct real valued multiple Wiener integrals for $\Phi^{\otimes n}$-valued functions (Subsection 5.1.1) and then

use them to study real valued nonlinear functionals of W (Subsection 5.1.2).

5.1.1 Multiple Wiener integrals for $\Phi^{\otimes n}$-valued functions

Throughout this subsection we will assume, unless otherwise stated, that $n \geq 1$ and $T = [0,T_0]$, $T_0 > 0$ are fixed but arbitrary. Denote by $L_Q(T^n \to \Phi^{\otimes n})$ the class of $\Phi^{\otimes n}$-valued measurable functions f on T^n such that

$$(5.1.1) \qquad \int_{T^n} Q^{\otimes n}(f(\underline{t}),f(\underline{t}))d\underline{t} < \infty \qquad \underline{t} = (t_1,\ldots,t_n)$$

where $Q^{\otimes n}$ is the c.p.d.b. form on $\Phi^{\otimes n} \times \Phi^{\otimes n}$ which is the n^{th} tensor product of Q (see Section 4.1.1).

We shall define multiple Wiener integrals for elements in the class $L_Q(T^n \to \Phi^{\otimes n})$.

A useful concept in the theory of finite dimensional multiple Wiener integrals is that of symmetric real valued functions on T^n (see Theorem 2.3.3 and (2.3.10) and (2.3.11)). We now introduce the analogous concept of symmetrization of $\Phi^{\otimes n}$-valued and $K^{\otimes n}$-valued multivariate functions, where K is a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$.

Definition 5.1.1. Let $n \geq 1$ and $f: \mathbb{R}_+^n \to K^{\otimes n}$. Denote by $\tilde{f}$ the symmetrization of f defined by

$$(5.1.2) \qquad \tilde{f}(\underline{t}) = \frac{1}{n!} \sum_{\Pi} f_{\Pi}(\underline{t})$$

where the sum is taken over all permutations $\Pi = (\Pi(1),\ldots,\Pi(n))$ of $(1,\ldots,n)$ and

$$(5.1.3)\qquad f_\pi(\underline{t}) = \sum_{j_1\ldots j_n=1}^{\infty} \{\langle f(t_{\Pi(1)},\ldots,t_{\Pi(n)}), e_{j_{\Pi(1)}} \otimes\cdots\otimes e_{j_{\Pi(n)}}\rangle_{K^{\otimes n}} \cdot e_{j_1}\otimes\cdots\otimes_{j_n}\}$$

for $\{e_i\}_{i\geq 1}$ a CONS of K.

Proposition 5.1.1. Let $(K_1, \langle\cdot,\cdot\rangle_{K_1})$ be another separable Hilbert space such that $K \subset K_1$. Then $f_\Pi(\underline{t})$ is well defined for all permutations $\Pi = (\Pi(1),\ldots,\Pi(n))$ and therefore independent of the CONS $\{e_i\}_{i\geq 1}$ in K.

Before giving the proof of the above proposition we shall define the symmetrization of a $\Phi^{\otimes n}$-function.

Corollary 5.1.1. Let $n \geq 1$ and $f:\mathbb{R}_+^n \to \Phi^{\otimes n}$. Then the *symmetrization* $\tilde{f}$ of f defined as the symmetrization of f on any of the Hilbert spaces $H_r^{\otimes n}$ $r \geq 1$ is well defined.

The *proof* of the corollary follows since $\Phi^{\otimes n} = \bigcap_{r=1}^{\infty} H_r^{\otimes n}$, $H_r^{\otimes n} \supset H_s^{\otimes n}$ for $s > r$ and by using Proposition 5.1.1.

Proof of Proposition 5.1.1 Suppose $K \subset K_1$ and let $\{\psi_i\}_{i\geq 1}$ and $\{e_i\}_{i\geq 1}$ be CONS for K_1 and K respectively and assume that for each $\underline{t} \in \mathbb{R}_+^n$ $f(\underline{t})$ is $K^{\otimes n}$ and $K_1^{\otimes n}$-valued. Then

$$f_\Pi^K(\underline{t}) = \sum_{j_1\ldots j_n=1}^{\infty} \langle f(\underline{t}_\Pi), e_{j_{\Pi(1)}}\otimes\cdots\otimes e_{j_{\Pi(n)}}\rangle_{K^{\otimes n}}\, e_{j_1}\otimes\cdots\otimes e_{j_n}$$

and

$$f_\Pi^{K_1}(\underline{t}) = \sum_{\ell_1\ldots\ell_n=1}^{\infty} \langle f(\underline{t}_\Pi), \psi_{\ell_{\Pi(1)}}\otimes\cdots\otimes \psi_{\ell_{\Pi(n)}}\rangle_{K_1^{\otimes n}}\, \psi_{\ell_1}\otimes\cdots\otimes \psi_{\ell_n}$$

Thus
$$f_\Pi^{K_1}(\underline{t}) = \sum_{\ell_1\ldots\ell_n=1}^{\infty}\ \sum_{j_1\ldots j_n=1}^{\infty} \{\langle f(\underline{t}_\Pi), e_{j_{\Pi(1)}}\otimes\cdots\otimes e_{j_{\Pi(n)}}\rangle_{K^{\otimes n}} \cdot \langle \psi_{\ell_{\Pi(1)}}\otimes\cdots\otimes\psi_{\ell_{\Pi(n)}}, e_{j_{\Pi(1)}}\otimes\cdots\otimes e_{j_{\Pi(n)}}\rangle_{K_1^{\otimes n}} \psi_{\ell_1}\otimes\cdots\otimes\psi_{\ell_n}\}\ .$$

But for all permutations Π

$$<\psi_{\ell_{\Pi(1)}} \otimes \cdots \otimes \psi_{\ell_{\Pi(n)}}, e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}}>_{K_1^{\otimes n}} = <\psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n},$$

$$e_{j_1} \otimes \cdots \otimes e_{j_n}>_{K_1^{\otimes n}}$$

then

$$f_\Pi^{K_1}(\underline{t}) = \sum_{j_1 \ldots j_n = 1}^{\infty} <f(\underline{t}_\Pi), e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}}>_{K^{\otimes n}} \cdot$$

$$\{ \sum_{\ell_1 \ldots \ell_n = 1}^{\infty} <e_{j_1} \otimes \cdots \otimes e_{j_n}, \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n}>_{K_1^{\otimes n}} \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n} \}$$

$$= \sum_{j_1 \ldots j_n = 1}^{\infty} <f(\underline{t}_\Pi), e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}}>_{K^{\otimes n}} e_{j_1} \otimes \cdots \otimes e_{j_n} = f_\Pi^K(\underline{t}).$$

The above argument proves that $f_\Pi(\underline{t})$ is independent of the CONS in K.

Q.E.D.

For $T = [0, T_0]$, $T_0 > 0$ or $T = \mathbb{R}_+$ we denote by $L^2(T^n \to K^{\otimes n})$ the Hilbert space of $K^{\otimes n}$-valued measurable functions on T^n such that

$$\int_{T^n} \| f(\underline{t}) \|^2_{K^{\otimes n}} d\underline{t} < \infty.$$

<u>Proposition 5.1.2</u>. Let $T = [0, T_0]$, $T_0 > 0$ or $T = \mathbb{R}_+$. For $f \in L^2(T^n \to K^{\otimes n})$ let

$$(Sf)(\underline{t}) = \tilde{f}(\underline{t}) = \frac{1}{n!} \sum_\Pi f_\Pi(\underline{t}) .$$

Then S is an orthogonal projection operator on $L^2(T^n \to K^{\otimes n})$ whose range S may be identified with the n-fold symmetric tensor product space $(L^2(T) \otimes K)^{\circledS n}$.

<u>Proof</u>. It is known (Reed and Simon (1980)) that $L^2(T^n \to K^{\otimes n}) \cong L^2(T^n) \otimes K^{\otimes n} \cong (L^2(T) \otimes K)^{\otimes n}$. Then it is enough to consider functions of the form

$$f(\underline{t}) = f(t_1, \ldots, t_n) = f_{i_1}(t_1) \ldots f_{i_n}(t_n) e_{i_1} \otimes \cdots \otimes e_{i_n}$$

where $\{f_i\}_{i \geq 1}$ and $\{e_i\}_{i \geq 1}$ are CONS for $L^2(T)$ and K respectively. Then since

for all Π

$$f_\Pi(\underline{t}) = f_{i_1}(t_{\Pi(1)})\dots f_{i_n}(t_{\Pi(n)}) \sum_{j_1\dots j_n=1}^{\infty} \{<e_{i_1}\otimes\dots\otimes e_{i_n},$$

$$e_{j_{\Pi(1)}}\otimes\dots\otimes e_{j_{\Pi(n)}}>_{K^{\otimes n}} e_{j_1}\otimes\dots\otimes e_{j_n}\}$$

$$= f_{i_{\Pi^{-1}(1)}}(t_1)\dots f_{i_{\Pi^{-1}(n)}}(t_n)\, e_{i_{\Pi^{-1}(1)}}\otimes\dots\otimes e_{i_{\Pi^{-1}(n)}} \quad \frac{1}{n!}\sum_{\Pi} f_\Pi(\underline{t})$$

$$= \frac{1}{n!}\sum_{\Pi}(f_{i_{\Pi(1)}}(t_1)\, e_{i_{\Pi(1)}})\otimes\dots\otimes(f_{i_{\Pi(n)}}(t_n)\, e_{\Pi(n)})$$

is an element of $(L^2(T)\otimes K)^{\otimes n}$.

Next for each permutation Π, from (5.1.3) it is seen that the operator $\mathcal{D}_\Pi(f) = f_\Pi$ is linear with adjoint equal to its inverse. Then

$$\mathcal{P}^* = \frac{1}{n!}\sum_{\Pi}\mathcal{D}^*_\Pi = \frac{1}{n!}\sum_{\Pi}\mathcal{D}_{\Pi^{-1}} = \mathcal{P}.$$

Also from (5.1.3) we have that for each permutation Π

$$(\mathcal{P}f)_\Pi = \frac{1}{n!}\sum_{\Pi^*}\sum_{j_1\dots j_n=1}^{\infty} <f(\underline{t}_{\Pi^*}), e_{j_{\Pi^*(1)}}\otimes\dots\otimes e_{j_{\Pi^*(n)}}>_{K^{\otimes n}} e_{j_1}\otimes\dots\otimes e_{j_n}$$

where the first sum is taken over all permutations Π^* of Π. Then for all permutations Π of $(1,\dots,n)$ $(\mathcal{P}f)_\Pi = (\mathcal{P}f)$ and hence

$$\mathcal{P}^2(f) = \mathcal{P}(\mathcal{P}(f)) = \frac{1}{n!}\sum_{\Pi}(\mathcal{P}f)_\Pi = \mathcal{P}f.$$

Thus $\mathcal{P}$ is a projection operator whose range can be identified with the symmetric tensor product space $(L^2(T)\otimes K)^{\circledS n}$.

Q.E.D.

Corollary 5.1.2 Let T, K and f be as in the last proposition. Then

$$\int_{T^n} \|\tilde{f}(\underline{t})\|^2_{K^{\otimes n}} d\underline{t} \le \int_{T^n} \| f(\underline{t})\|^2_{K^{\otimes n}}\, d\underline{t}\ .$$

The proof follows by using the fact that $\mathcal{P}f = \tilde{f}$ is a projection operator

on $L^2(T^n \to K^{\otimes n})$.

<u>Remark</u> Multiple Wiener integrals on a Hilbert space have been defined by Miyahara (1981) for the case of a cylindrical Brownian motion on H_o. In his case $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$ and the CONS $\{\phi_j\}_{j\geq 1}$ of eigenvectors of L diagonalizes Q (see Example 4.1.7). In our case we do not consider a cylindrical Brownian motion but rather a Φ'-valued Wiener process with an H_{-q} continuous version for $q \geq r_1 + r_2$. Moreover, the c.p.d.b. form Q on $\Phi \times \Phi$ is not assumed to be diagonalized by $\{\phi_j\}_{j\geq 1}$. This leads to finite dimensional multiple integrals with dependent integrators of the type studied in Chapters II and III (see also Lemma 4.1.5).

In order to define real valued multiple Wiener integrals for elements in $L_Q(T^n \to \Phi^{\otimes n})$ we shall first construct multiple integrals for $H_q^{\otimes n}$-valued functions for $q \geq r_1 + r_2$ (as in the case of stochastic integrals in Chapter IV) and then apply density arguments to define them on the spaces $L_Q(T^n \to \Phi^{\otimes n})$ and $L^2(T^n \to H_Q^{\otimes n})$.

<u>Multiple Wiener integrals for elements in $L^2(T^n \to H_q^{\otimes n})$</u>

<u>Definition 5.1.2</u> Let $n \geq 1$ fixed and $q \geq r_1 + r_2$. For $f \in L^2(T^n \to H_q^{\otimes n})$ define the real valued multiple Wiener integral of f with respect to the Φ'-valued Wiener process W_t by

$$(5.1.4) \qquad I_{n,T}(f) = \sum_{j_1 \ldots j_n = 1}^{\infty} \int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{j_i}](\underline{t})$$

where $\{e_i\}_{i\geq 1}$ is a CONS for H_q and each multiple integral in the RHS of (5.1.4) is an integral with respect to the symmetric tensor product measure $\bigotimes_{i=1}^{n} W[e_i]$ defined in Sections 2.3 and 3.1 (see also Lemma 4.1.5).

<u>Proposition 5.1.3</u> Let H be the linear (Hilbert) space of $(W_t)_{t\in \mathbb{R}_+}$ defined

in (4.1.25). Let $n \geq 1$, $q \geq r_1 + r_2$ and $f \in L^2(T^n \to H_q^{\otimes n})$. Then the multiple integral (5.1.4) is a well defined element in $\mathcal{H}^{\otimes n}$ (the n-fold symmetric tensor product of $\mathcal{H}$) and of $L^2(\Omega, \mathcal{F}^W, P)$.

<u>Proof</u> Let $\{e_i\}_{i\geq 1}$ be a CONS for H_q. Then for each $j_1, \ldots j_n$

$$\int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle^2_{q^{\otimes n}} \, d\underline{t} \leq \int_{T^n} \| f(\underline{t}) \|^2_{q^{\otimes n}} \, d\underline{t} < \infty$$

and by Lemma 4.1.5 and Theorem 2.3.2 $\langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}}$ is $\overset{n}{\underset{i=1}{\otimes}} W[e_{j_i}]$-integrable and each integral in the RHS of (5.1.4) is an element in $\mathcal{H}^{\otimes n}$. Next by Theorem 2.3.3 (b)

$$(5.1.5) \quad E\left(\int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \overset{n}{\underset{i=1}{\otimes}} W[e_{j_i}](\underline{t})\right)^2$$

$$\leq \int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle^2_{q^{\otimes n}} Q(e_{j_1}, e_{j_1}) \ldots Q(e_{j_n}, e_{j_n}) d\underline{t}$$

$$= \int_{T^n} Q^{\otimes n}(\langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \ldots \otimes e_{j_n}, \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} \cdot e_{j_1} \otimes \ldots \otimes e_{j_n}) d\underline{t} \,.$$

Then if $\{e_j = (1+\lambda_j)^{-q} \phi_j\}_{j\geq 1}$ is the CONS in H_q, by applying Cauchy-Schwartz inequality and (5.1.5) above we have

$$(5.1.6) \quad E(I_{n,T}(f))^2 \leq \left\{ \sum_{j_1 \ldots j_n = 1}^{\infty} (1+\lambda_{j_1})^{-2(q-r_2)} \ldots (1+\lambda_{j_n})^{-2(q-r_2)} \right\} \cdot$$

$$\left\{ \sum_{j_1 \ldots j_n = 1}^{\infty} (1+\lambda_{j_1})^{2r_2} \ldots (1+\lambda_{j_n})^{2r_2} \cdot \right.$$

$$\left. E\left(\int_{T^n} \langle f(\underline{t}), \phi_{j_1} \otimes \ldots \otimes \phi_{j_n} \rangle_{q^{\otimes n}} \, d \overset{n}{\underset{i=1}{\otimes}} W[e_{j_i}](\underline{t})\right)^2 \right\}$$

$$\leq \theta_1^n \sum_{j_1 \ldots j_n = 1}^{\infty} \prod_{i=1}^{n} (1+\lambda_{j_i})^{2r_2} \int_{T^n} Q^{\otimes n}(\langle f(\underline{t}), \phi_{j_1} \otimes \ldots \otimes \phi_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \ldots \otimes e_{j_n}, \langle f(\underline{t}), \phi_{j_1} \otimes \ldots \otimes \phi_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \ldots \otimes e_{j_n}) d\underline{t}$$

(Proposition 4.1.2)

$$\leq \theta_1^n \theta_2^n \sum_{j_1 \ldots j_n = 1}^{\infty} \prod_{i=1}^{n} (1+\lambda_{j_i})^{2r_2} \| e_{j_1} \otimes \ldots \otimes e_{j_n} \|^2_{q^{\otimes n}} \int_{T^n} \langle f(\underline{t}), \phi_{j_1} \otimes \ldots \otimes \phi_{j_n} \rangle^2_{q^{\otimes n}} d\underline{t}$$

$$= \theta_1^n \theta_2^n \int_{T^n} \sum_{j_1 \ldots j_n = 1}^{\infty} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle^2_{q^{\otimes n}} d\underline{t} \qquad (\| e_{j_1} \otimes \ldots \otimes e_{j_n} \|_{q^{\otimes n}} = 1)$$

$$= \theta_1^n \theta_2^n \int_{T^n} \| f(\underline{t}) \|^2_{q^{\otimes n}} d\underline{t} < \infty \ .$$

Therefore $E(I_{n,T}(f))^2 < \infty$ and the multiple series (5.1.4) converges in mean square. Then the linearity of $I_{n,T}(\cdot)$ follows.

The next step will also show that $I_{n,T}(\cdot)$ is independent of the choice of the CONS in H_q.

Let $q_1 \geq r_1 + r_2$, $q \geq q_1$ and assume that f belongs to $L^2(T^n \to H_{q_1}^{\otimes n})$ and $\{\psi_i\}_{i \geq 1}$ is a CONS in H_{q_1}. Then using Lemma 4.1.5, the bilinearity of $Q^{\otimes n}$ in $H_q^{\otimes n} \times H_q^{\otimes n}$ and Corollary 5.1.2, we have that for all $m \geq 1$

$$(5.1.7) \qquad E\Big(\sum_{j_1 \ldots j_n = 1}^{m} \int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{j_i}](\underline{t})$$

$$- \sum_{j_1 \ldots j_n = 1}^{m} \int_{T^n} \langle f(\underline{t}), \psi_{j_1} \otimes \ldots \otimes \psi_{j_n} \rangle_{q_1^{\otimes n}} d \bigotimes_{i=1}^{n} W[\psi_{j_i}](\underline{t}) \Big)^2$$

$$\leq \int_{T^n} Q^{\otimes n} \Big(\sum_{j_1 \ldots j_n = 1}^{m} (\langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} e_{j_i} \otimes \ldots \otimes e_{j_n}$$

$$- \langle f(\underline{t}), \psi_{j_1} \otimes \ldots \otimes \psi_{j_n} \rangle_{q_1^{\otimes n}} \psi_{j_1} \otimes \ldots \otimes \psi_{j_n}),$$

$$\sum_{j_1 \ldots j_n = 1}^{m} (\langle f(\underline{t}), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \ldots \otimes e_{j_n}$$

$$- \langle f(\underline{t}), \psi_{j_1} \otimes \ldots \otimes \psi_{j_n} \rangle_{q_1^{\otimes n}} \psi_{j_1} \otimes \ldots \otimes \psi_{j_n})\Big) d\underline{t}$$

$$\leq \theta_2^n \int_{T^n} \| \sum_{j_1 \ldots j_n = 1}^{m} \langle f(\underline{t}), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \cdots \otimes e_{j_n}$$

$$- \sum_{j_1 \ldots j_n = 1}^{m} \langle f(\underline{t}), \psi_{j_1} \otimes \cdots \otimes \psi_{j_n} \rangle_{q_1^{\otimes n}} \|_q^2 d\underline{t}$$

which goes to zero as $m \to \infty$ by dominated convergence theorem since

$$\| \sum_{j_1 \ldots j_n = 1}^{m} \langle f(\underline{t}), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle_{q^{\otimes n}} e_{j_1} \otimes \cdots \otimes e_{j_n}$$

$$- \sum_{j_1 \ldots j_n = 1}^{m} \langle f(\underline{t}), \psi_{j_1} \otimes \cdots \otimes \psi_{j_n} \rangle_{q_1^{\otimes n}} \psi_{j_1} \otimes \cdots \otimes \psi_{j_n} \|_q^2$$

$$\leq 2(\| f(\underline{t}) \|^2_{q_1^{\otimes n}} + \| f(\underline{t}) \|^2_{q^{\otimes n}}) \qquad \text{all } m \geq 1.$$

Then the proof of the proposition is complete. Q.E.D.

The multiple integral $I_{n,T}(\cdot)$ has several properties analogous to those of the multiple Wiener integral for a real valued Wiener process of Itô (1951). We now present them.

<u>Proposition 5.1.4</u> Let $n \geq 1$, $q \geq r_1 + r_2$ and $f \in L^2(T^n \to H_q^{\otimes n})$. Then if $\tilde{f}$ denotes the symmetrization of f (Definition 5.1.1).

a) $I_{n,T}(\tilde{f}) = I_{n,T}(f)$.

b) $E(I_{n,T}(f)) = 0$.

c) $E(I_{n,T}(f))^2 \leq n!\ \theta_2^n \| f \|^2_{L^2(T^n \to H_q^{\otimes n})} < \infty$.

d) If $g \in L^2(T^m \to H_q^{\otimes m})$ for $m \geq 1$, then

$$E(I_{n,T}(f) I_{m,T}(g)) = \delta_{nm}\ n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^n \to H_Q^{\otimes n})}.$$

e) $E(I_{n,T}(f))^2 = E(I_{n,T}(\tilde{f}))^2 = n! \| \tilde{f} \|^2_{L^2(T^n \to H_Q^{\otimes n})} \leq n! \| f \|^2_{L^2(T^n \to H_Q^{\otimes n})}$.

<u>Proof</u> a) Since for all $j_1 \dots j_n$ and $\underline{t} \in T$

$$(5.1.8) \qquad \langle \tilde{f}(\underline{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}} = \frac{1}{n!} \sum_{\Pi} \sum_{k_1 \dots k_n = 1}^{\infty} \cdot$$

$$\{\langle f(\underline{t}_\Pi), e_{k_{\Pi(1)}} \otimes \dots \otimes e_{k_{\Pi(n)}} \rangle_{q^{\otimes n}} \langle e_{k_{\Pi(1)}} \otimes \dots \otimes e_{k_{\Pi(n)}}, e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}}\}$$

$$= \langle f(\underline{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}}$$

it follows from (5.1.4) that $I_{n,T}(\tilde{f}) = I_{n,T}(f)$.

b) By Lemma 4.1.5, for each $j_1 \dots j_n$

$$E\left(\int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{j_i}](\underline{t})\right) = 0$$

and hence $E(I_{n,T}(f)) = 0$.

d)

$$(5.1.9) \qquad E(I_{n,T}(f) I_{m,T}(g)) = \sum_{j_1 \dots j_n = 1}^{\infty} \sum_{k_1 \dots k_n = 1}^{\infty} \cdot$$

$$\{E\left(\int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{j_i}](\underline{t})\right.$$

$$\left.\int_{T^n} \langle g(\underline{t}), e_{k_1} \otimes \dots \otimes e_{k_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{m} W[e_{k_i}](\underline{t})\right)\}.$$

Then by Lemma 4.1.5 $E(I_{n,T}(f) I_{m,T}(g)) = 0$ if $n \neq m$ and if $n = m$

$$E\left(\int_{T^n} \langle f(\underline{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{j_i}](\underline{t}) \cdot\right.$$

$$\left.\int_{T^n} \langle g(\underline{t}), e_{k_1} \otimes \dots \otimes e_{k_n} \rangle_{q^{\otimes n}} \, d \bigotimes_{i=1}^{n} W[e_{k_i}](\underline{t})\right)$$

$$= \int_{T^n} Q^{\otimes n} \left(\frac{1}{n!} \sum_{\Pi} \langle f(\underline{t}_\Pi), e_{j_{\Pi(1)}} \otimes \dots \otimes e_{j_{\Pi(n)}} \rangle_{q^{\otimes n}} e_{j_1} \otimes \dots \otimes e_{j_n},\right.$$

$$\left.\frac{1}{n!} \sum_{\Pi} \langle f(\underline{t}_\Pi), e_{j_{\Pi(1)}} \otimes \dots \otimes e_{j_{\Pi(n)}} \rangle_{q^{\otimes n}} e_{j_1} \otimes \dots \otimes e_{j_n}\right) d\underline{t} \, .$$

Therefore, using the continuity of $Q^{\otimes n}$ on $H_q^{\otimes n} \times H_q^{\otimes n}$

$$E(I_{n,T}(f)I_{n,T}(g)) = \int_{T^n} Q^{\otimes n}(\tilde{f}(\underline{t}),\tilde{g}(\underline{t}))d\underline{t}$$

$$= \langle\tilde{f},\tilde{g}\rangle_{L^2(T^n \to H_Q^{\otimes n})}$$

and (d) is proved.

The proof of (e) follows from (d), (a) and Corollary 5.1.2. The proof of (c) follows from (d) and since for $\psi \in H_q^{\otimes n}$

$$\|\psi\|^2_{Q^{\otimes n}} \le \theta_2^n \|\psi\|^2_{q^{\otimes n}} \qquad q \ge r_1 + r_2.$$

Q.E.D.

We will extend the definition of $I_{n,T}(\cdot)$ to functions in $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$, denoted by I_n, and show (Corollary 5.1.4) that orthogonal series of I_n are dense in $L^2(\Omega,F^W,P)$.

Multiple Wiener integrals for elements in $L^2(T^n \to H_Q^{\otimes n})$

Lemma 5.1.1 Let $f \in L^2(T^n \to H_Q^{\otimes n})$. Then for all $q \ge r_1 + r_2$ there exists a sequence $\{f_m\}_{m\ge 1}$, $f_m \in L^2(T^n \to H_q^{\otimes n})$ $m \ge 1$, such that $f_m \underset{m\to\infty}{\to} f$ in $L^2(T^n \to H_Q^{\otimes n})$.

Proof Let $f \in L^2(T^n \to H_Q^{\otimes n})$, then given $\varepsilon > 0$ there exists an $H_Q^{\otimes n}$-valued step function g^ε such that

$$\int_{T^n} \| f(\underline{t}) - g^\varepsilon(\underline{t}) \|^2_{Q^{\otimes n}} d\underline{t} < \varepsilon/2 \tag{5.1.10}$$

where $g^\varepsilon(\underline{t}) = \sum_{i=1}^k a_i 1_{A_i}(\underline{t})$, $a_i \in H_Q^{\otimes n}$, $A_i \in \mathcal{B}(T^n)$ and $m^{\otimes n}(A_i) < \infty$ $i=1,\dots,k$ some k, $m^{\otimes n}$ denoting the Lebesgue measure on $(T^n,\mathcal{B}(T^n))$.

Next, since for $q \ge r_1 + r_2$ $H_q^{\otimes n}$ is dense in $H_Q^{\otimes n}$, then there exist $b_i \in H_q^{\otimes n}$ $i=1,\dots,k$ such that

$$\| a_i - b_i \|^2_{Q^{\otimes n}} < \frac{\varepsilon}{2km^{\otimes n}(A_i)} .$$

Define $\quad f^{\varepsilon}(\underline{t}) = \sum_{i=1}^{k} b_i \, 1_{A_i}(\underline{t}),$ then $f^{\varepsilon} \in L^2(T^n \to H_q^{\otimes n})$

and

$$\int_{T^n} \| f^{\varepsilon}(\underline{t}) - g^{\varepsilon}(\underline{t}) \|^2_{Q^{\otimes n}} d\underline{t} < \varepsilon/2 .$$

Then from the last expression and (5.1.10)

$$\int_{T^n} \| f(\underline{t}) - f^{\varepsilon}(\underline{t}) \|^2_{Q^{\otimes n}} d\underline{t} < \varepsilon \tag{5.1.11}$$

and the existence of the required sequence follows.

Q.E.D.

Note that the above result holds if $T = \mathbb{R}_+$.

Definition 5.1.3 By Propositions 5.1.3 and 5.1.4 we have that $I_{n,T}$ defined in (5.1.4) is a bounded linear operator from $L^2(T^n \to H_q^{\otimes n})$ to $L^2(\Omega)$ (in fact to $\mathcal{H}^{\otimes n}$). *Hence by Proposition 5.1.4 (e) and Lemma 5.1.1,* $I_{n,T}(\cdot)$ has a unique extension to $L^2(T^n \to H_Q^{\otimes n})$. We denote this extension by $I_{n,T}$ and call it the n^{th} real valued multiple Wiener integral for elements in $L^2(T^n \to H_Q^{\otimes n})$.

We summarize the main properties of $I_{n,T}$ in the following lemma whose proof follows by Proposition 5.1.3 and 5.1.4 and the above definition.

Lemma 5.1.2 Let $f \in L^2(T^n \to H_Q^{\otimes n})$. Then if $\tilde{f}$ denotes the symmetrization of f on $H_Q^{\otimes n}$ (see Definition 5.1.1)

a) $I_{n,T}(\tilde{f}) = I_{n,T}(f) \in \mathcal{H}^{\otimes n}$.

b) $E(I_{n,T}(f)) = 0$.

c) If $g \in L^2(T^m \to H_Q^{\otimes m})$ for $m \geq 1$ then

$$E(I_{n,T}(f) I_{m,T}(g)) = \delta_{nm} \, n! \int_{T^n} Q^{\otimes n}(\tilde{f}(\underline{t}), \tilde{g}(\underline{t})) d\underline{t} .$$

d) $E(I_{n,T}(f))^2 = n! \int_{T^n} Q^{\otimes n}(\tilde{f}(\underline{t}), \tilde{f}(\underline{t})) d\underline{t} \leq n! \int_{T^n} Q^{\otimes n}(f(\underline{t}), f(\underline{t})) d\underline{t}$.

The proof follows by Definition 5.1.3 and Propositions 5.1.3 and 5.1.4.

Iterated stochastic integrals We now define the real valued iterated stochastic integral $J_n(\cdot)$ for elements in $L^2(\mathbb{R}^n \to H_q^{\otimes n})$ $q \geq r_1 + r_2$, and show its relationship with the multiple Wiener integral $I_{n,T}(\cdot)$, $T = [0,t]$ $t \geq 0$, of Definition 5.1.2 for elements in $L^2(T^n \to H_q^{\otimes n})$. This connection will allow us to extend the definition of $I_{n,T}(\cdot)$ to functions in the space $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$.

In what follows $L^2(\mathbb{R}_+^n) = L^2(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}^n), d\underline{t})$ where $d\underline{t}$ stands for the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

Let $\{f_k\}_{k\geq 1}$ and $\{e_k\}_{k\geq 1}$ be complete orthonormal sets for $L^2(\mathbb{R}_+)$ and H_q respectively where $q \geq r_1 + r_2$. Then (see Reed and Simon (1980))

$$(5.1.12) \qquad \left\{ f_{\ell_1}(t_1) \ldots f_{\ell_n}(t_n) e_{k_1} \otimes \ldots \otimes e_{k_n} \quad \begin{matrix} \ell_1 \geq 1, \ldots, \ell_n \geq 1 \\ k_1 \geq 1, \ldots, k_n \geq 1 \end{matrix} \right\}$$

is a CONS for $L^2(\mathbb{R}^n) \otimes H_q^{\otimes n} \cong L^2(T^n \to H_q^{\otimes n})$.

For $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$ and n fixed, let

$$(5.1.13) \qquad f(\underline{t}) = (f_{\ell_1}(t_1) e_{k_1}) \otimes \ldots \otimes (f_{\ell_n}(t_n) e_{k_n}) \qquad \underline{t} = (t_1, \ldots, t_n)$$

and for $t > 0$ define

$$J_o(f)_{t_1} = f_{\ell_1}(t_1) e_{k_1}$$

$$(5.1.14) \qquad J_{m-1}(f)_{t_m} = \int_o^{t_m} \langle J_{m-2}(f)_{t_{m-1}}, dW_{t_{m-1}} \rangle_q \; f_{\ell_m}(t_m) e_{k_m} \qquad 2 \leq m \leq n$$

$$(5.1.15) \qquad J_{n,t}(f) = \int_o^t \langle J_{n-1}(f)_{t_n}, dW_{t_n} \rangle_q$$

where the stochastic integrals of the RHS are in the sense of Definition 4.2.2. If f and g are two $H_q^{\otimes n}$-valued functions as in (5.1.13) define $J_{n,t}(f+g) = J_{n,t}(f) + J_{n,t}(g)$. Then $J_{n,t}$ is extended to the linear mani-

fold S_n generated by $H_q^{\otimes n}$-valued functions of the form (5.1.13).

Proposition 5.1.5 If $f \in S_n$, then for each $t > 0$ $J_{n,t}(f)$ is well defined and

$$E(J_{n,t}(f))^2 \leq \int_{\mathbb{R}_+^n} \| f(\underline{t}) \|^2_{Q^{\otimes n}} d\underline{t} \leq \int_{\mathbb{R}_+^n} \| f(\underline{t}) \|^2_{q^{\otimes n}} d\underline{t} < \infty .$$

Proof By the preceding paragraph to this proposition, it is enough to prove the result for functions of the form (5.1.13). We first show that for $m = 1,\ldots,n$ $J_{m-1}(f)_{t_m}$ belongs to M_q (see Definitions 4.2.1 and 4.2.2). Using Fubini's theorem

$$\prod_{i=1}^{n} \int_{\mathbb{R}_+} \| f_{\ell_i}(t_i) e_{k_i} \|^2_Q \, dt_i = \int_{\mathbb{R}_+^n} \| f_{\ell_1}(t_1) e_{k_1} \otimes \cdots \otimes f_{\ell_n}(t_n) e_{k_n} \|^2_{Q^{\otimes n}} d\underline{t}$$

$$= \int_{\mathbb{R}_+^n} \| f(\underline{t}) \|^2_{Q^{\otimes n}} d\underline{t} \leq \int_{\mathbb{R}_+^n} \| f(\underline{t}) \|^2_{q^{\otimes n}} d\underline{t} < \infty$$

and therefore for each $1 \leq m \leq n$

$$\int_{\mathbb{R}_+^m} \prod_{i=1}^{m} \| f_{\ell_i}(t_i) e_{k_i} \|^2_Q \, dt_1 \ldots dt_m < \infty \qquad \text{a.e. } dt_{m+1} \ldots dt_n .$$

Then $J_o(f)_{t_1} \in M_q$ and $\int_o^{t_2} \langle J_o(f)_{t_1}, dW_{t_1} \rangle_q$ is given by Definition 4.2.2. Therefore $J_1(f)_{t_2}$ is H_q-valued and from Propositions 4.2.1 and 4.2.2 it is non-anticipative such that for each $t_3 > 0$

$$E \int_o^{t_3} \| J_1(f)_{t_2} \|^2_Q \, dt_2 = \int_o^{t_3} \int_o^{t_2} \prod_{i=1}^{2} \| f_{\ell_i}(t_i) e_{k_i} \|^2_Q \, dt_1 dt_2 < \infty .$$

Proceeding in the same way we have that for $2 \leq m < n$, $J_{m_1}(f)_{t_m}$ is H_q-valued, non-anticipative and

$$E \int_o^{\infty} \| J_{m-1}(f)_{t_m} \|^2_Q \, dt_m = \int_o^{\infty} \int_o^{t_{m-1}} \cdots \int_o^{t_2} \prod_{i=1}^{m} \| f_{\ell_i}(t_i) e_{k_i} \|^2_Q \, dt_1 \ldots dt_m < \infty$$

i.e., $J_{m-1}(f)_{t_m} \in M_q$ and hence the stochastic integral $\int_o^t \langle J_{m-1}(f)_{t_m}, dW_{t_m} \rangle_q$

can be defined as in Definition 4.2.2. Thus for each $t > 0$ $J_{n,t}(f)$ is well defined and

$$E(J_{n,t}(f))^2 = \int_0^t E\| J_{n-1}(f)_{t_n} \|^2_{H_Q} dt =$$

$$\int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} \prod_{i=1}^n \| f_{\ell_i}(t_i) e_{k_i} \|^2_Q \, dt_1 \ldots dt_n \le$$

$$\int_{\mathbb{R}_+^n} \| f(\underline{u}) \|^2_{Q^{\otimes n}} \, d\underline{u} \le \int_{\mathbb{R}_+^n} \| f(\underline{u}) \|^2_{q^{\otimes n}} \, d\underline{u}$$

Q.E.D.

<u>Definition 5.1.4</u> Since elements of the form (5.1.13) generate $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ and from Proposition 5.1.5 for each $t > 0$ $J_{n,t}(\cdot)$ is a bounded linear transformation from S_n to $L^2(\Omega, F^W, P)$, then it can be extended to $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$. This extension is also denoted by $J_{n,t}(\cdot)$ and called the n^{th} <u>iterated stochastic integral</u>. Moreover, using Lemma 5.1.1 $J_{n,t}(\cdot)$ is also extended to $L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$.

We now present the relation between $J_{n,t}(\cdot)$ and $I_{n,t}(\cdot)$.

<u>Proposition 5.1.6</u> Let $q \ge r_1 + r_2$ and $n \ge 1$. For any $t > 0$ let $T = [0,t]$ and $I_{n,T}(\cdot)$ be the multiple Wiener integral of Definition 5.1.2 for elements in $L^2(T^n \to H_q^{\otimes n})$. Let $f \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$, then for each $t > 0$ if f is restricted to $T = [0,t]$

(5.1.16) $$I_{n,T}(f) = I_{n,T}(\tilde{f}) = n! \, J_{n,t}(\tilde{f})$$

and

(5.1.17) $$I_{n,T}(f) = \int_0^t \langle g(s), dW_s \rangle_Q$$

where $g \in M_Q$ and the RHS of (5.1.17) is the stochastic integral of Definition 4.2.3. Moreover, $E \int_0^\infty \| g(s) \|^2_Q \, ds < \infty$.

<u>Proof</u> Let $\{f_k\}_{k \ge 1}$ and $\{e_k\}_{k \ge 1}$ be CONS for $L^2(\mathbb{R}_+)$ and H_q respectively

and $f(\underline{t})$ be as in (5.1.3). Then from Definition 5.1.1

$$\tilde{f}(\underline{t}) = \frac{1}{n!} \sum_{\Pi} (f_{\ell_{\Pi_1}}(t_{\Pi_1}) e_{k_{\Pi_1}}) \otimes \cdots \otimes (f_{\ell_{\Pi_n}}(t_{\Pi_n}) e_{k_{\Pi_n}})$$

and since for each $t > 0$ $J_{n,t}(f)$ and $I_{n,T}(f)$ are linear on f

$$J_{n,t}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} J_{n,t}((f_{\ell_{\Pi_1}}(t_{\Pi_1}) e_{k_{\Pi_1}}) \otimes \cdots \otimes (f_{\ell_{\Pi_n}}(t_{\Pi_n}) e_{k_{\Pi_n}}))$$

and

$$I_{n,T}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} I_{n,t}((f_{\ell_{\Pi_1}}(t_{\Pi_1}) e_{k_{\Pi_1}}) \otimes \cdots \otimes (f_{\ell_{\Pi_n}}(t_{\Pi_n}) e_{k_{\Pi_n}})). \tag{5.1.18}$$

But from (5.1.4) in Definition 5.1.2 and Lemma 2.3.1 we have that for all permutations $\Pi = (\Pi_1, \ldots, \Pi_n)$ of $(1, \ldots, n)$ and $T = [0,t]$, $t > 0$

$$I_{n,T}((f_{\ell_{\Pi_1}}(t_{\Pi_1}) e_{k_{\Pi_1}}) \otimes \cdots \otimes (f_{\ell_{\Pi_n}}(t_{\Pi_n}) e_{k_{\Pi_n}}))$$

$$= \int_{T^n} f_{\ell_{\Pi_1}}(t_{\Pi_1}) \cdots f_{\ell_{\Pi_n}}(t_{\Pi_n}) \, d \bigotimes_{i=1}^{n} W[e_{k_{\Pi_i}}](\underline{t})$$

$$= I_{W[e_{k_{\Pi_1}}]}(f_{\ell_{\Pi_1}}) \otimes \cdots \otimes I_{W[e_{k_{\Pi_n}}]}(f_{\ell_{\Pi_n}})$$

where $I_{W[e_\ell]}(f_\ell)$ is the isometric integral of f_ℓ w.r.t. the o.s.m. $W[e_\ell]$ (see Lemma 4.1.5 and Theorem 2.1.1). Then from (5.1.18)

$$I_{n,T}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} I_{W[e_{k_{\Pi_1}}]}(f_{\ell_{\Pi_1}}) \otimes \cdots \otimes I_{W[e_{k_{\Pi_n}}]}(f_{\ell_{\Pi_n}}) \tag{5.1.19}$$

$$= I_{W[e_{k_1}]}(f_{\ell_1}) \otimes \cdots \otimes I_{W[e_{k_n}]}(f_{\ell_n}).$$

On the other hand, from the definition of $J_{n,t}$ and (4.2.2) in Definition 4.2.2, for all $t > 0$

$$J_{n,t}((f_{\ell_{\Pi_1}}(t_{\Pi_1}) e_{k_{\Pi_1}}) \otimes \cdots \otimes (f_{\ell_{\Pi_n}}(t_{\Pi_n}) e_{k_{\Pi_n}}))$$

$$= \int_0^t \int_0^{t_{\Pi_{n-1}}} \cdots \int_0^{t_{\Pi_1}} f_{\ell_{\Pi_1}}(t_{\Pi_1}) \cdots f_{\ell_{\Pi_n}}(t_{\Pi_n}) \, dW_{t_{\Pi_1}}[e_{k_{\Pi_1}}] \cdots dW_{t_{\Pi_n}}[e_{k_{\Pi_n}}].$$

By the last expression, (5.1.19) and Theorem 3.3.3 it follows that if f is as in (5.1.13) then for each $t > 0$ $I_{n,T}(\tilde{f}) = n!\, J_{n,t}(\tilde{f})$. Next, since for each $t > 0$ $I_{n,T}(\cdot)$ and $J_{n,t}(\cdot)$ are bounded linear operators from $L^2(\mathbb{R}^n \to H_q^{\otimes n})$ to $L^2(\Omega)$, that agree (up to a constant n!) on a dense linear manifold S_n of $L^2(\mathbb{R}^n \to H_q^{\otimes n})$, then (5.1.16) follows for all f in $L^2(\mathbb{R}^n \to H_q^{\otimes n})$.

Now we shall prove (5.1.17). Let f be as in (5.1.13) and $g(s,\omega) = n!\, J_{n-1}(f)_s(\omega)$ $s \in \mathbb{R}_+$, $\omega \in \Omega$. In the proof of Proposition 5.1.5 we have shown that $J_{n-1}(f)$, defined in (5.1.14), belongs to M_q and moreover

$$E \int_0^\infty \| J_{n-1}(f)_s \|_Q^2 \, ds < \infty \, .$$

Therefore $g \in M_q$, $E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty$ and from (5.1.15) and (5.1.16) for each $t > 0$, $T = [0,t]$

$$I_{n,T}(f) = \int_0^t \langle g_s, dW_s \rangle_q \, .$$

The above result extends if f belongs to the linear manifold S_n.

Next, if $f \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$, there exists a sequence of functions $\{f_m\}_{m\geq 1}$ in S_n such that f_m converges to f in $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$. Then there exists a sequence of functions $\{g_m\}_{m\geq 1}$ in M_q such that

$$(5.1.20) \qquad E \int_0^\infty \| g_m(s) \|_Q^2 \, ds < \infty \qquad m \geq 1$$

and for each $t > 0$

$$(5.1.21) \qquad I_{n,t}(f_m) = \int_0^t \langle g_m(s), dW_s \rangle_q \, .$$

Then by Proposition 4.2.1 (d) for each $t > 0$

$$E(I_{n,T}(f_m - f_k))^2 = E\left(\int_0^t \langle g_m(s) - g_k(s), dW_s \rangle_q \right)^2$$
$$= E \int_0^t \| g_m(s) - g_k(s) \|_Q^2 \, ds \, .$$

On the other hand, by Proposition 5.1.4 (c), for each $t > 0$, $T = [0,t]$

$$(5.1.22) \qquad E(I_{n,T}(f_m - f_k))^2 \le n!\ \theta_2^n\ \| f_m - f_k \|^2_{L^2(T^n \to H_q^{\otimes n})}$$

$$\le n!\ \theta_2\ \| f_m - f_k \|^2_{L^2(\mathbb{R}_+^n \to H_q^{\otimes n})} \underset{m,k\to\infty}{\to} 0 .$$

Then

$$\sup_{0 \le t < \infty} E(I_{n,T}(f_m - f_k))^2 \to 0 \quad \text{as } m,k \to \infty$$

and

$$\sup_{0 \le t < \infty} E \int_0^t \| g_m(s) - g_k(s) \|_Q^2\, ds \to 0 \quad \text{as } m,k \to \infty .$$

Thus, using (5.1.20) and dominated convergence theorem

$$E \int_0^\infty \| g_m(s) - g_k(s) \|_Q^2\, ds = \sup_{0 \le t < \infty} E \int_0^t \| g_m(s) - g_k(s) \|_Q^2\, ds$$

$$\to 0 \quad \text{as } m,k \to \infty$$

and then there exists an H_q-valued, $\Omega \times \mathbb{R}_+$ measurable function g, $g \in M_Q$ such that

$$E \int_0^\infty \| g_m(s) - g(s) \|_Q^2\, ds \to 0 \quad \text{as } m \to \infty .$$

Thus by Definition 4.2.3, for each $t > 0$

$$E\Big(\int_0^t \langle g(s), dW_s \rangle_Q - \int_0^t \langle g_m(s), dW_s \rangle_q \Big)^2 \to 0 \quad \text{as } m \to \infty$$

and by (5.1.21) since for each $t > 0$

$$E(I_{n,T}(f) - I_{n,T}(f_m))^2 \to 0 \quad \text{as } m \to \infty$$

then

$$I_{n,T}(f) = \int_0^t \langle g(s), dW_s \rangle_Q \quad \text{for each } t > 0.$$

Q.E.D.

<u>Corollary 5.1.3</u> Let $f \in L(\mathbb{R}_+^n \to H_q^{\otimes n})$. Then if H is the Gaussian space of W_t (see 4.1.25)),

a) For each $t > 0$ $\quad J_{n,t}(\tilde{f}) \in H^{\otimes n}$.

b) $(J_{n,t}(\tilde{f}), F_t^W)_{t \in \mathbb{R}_+}$ is a square integrable martingale with a continuous modification and increasing process

$$E \int_0^t \| g(s) \|_Q^2 \, ds = E \int_{T^n} \| \tilde{f}(\underline{u}) \|_Q^2 \, d\underline{u} \qquad T = [0,t]$$

for some function $g \in M_Q$ with $E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty$.

The proof follows from the last proposition, Definition 4.2.3 and Proposition 4.2.2.

Remark Using Lemma 5.5.1 one can show that Propositions 5.1.6 abd Corollary 5.1.3 hold if f belongs to $L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$. However, we have left them the way they are to show the role played by the bilinear form Q and the Hilbert space H_Q.

Multiple Wiener integrals for elements in $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$.

Definition 5.1.5 In Definition 5.1.2 we have given the multiple Wiener integral $I_{n,T}(\cdot)$ for elements in the space $L^2(T^n \to H_q^{\otimes n})$ where $q \geq r_1 + r_2$ and $T = [0,t]$ $t > 0$ is a finite interval of the real line. Now let $f \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ and $I_{n,T}(f)$ be the multiple integral of f restricted to $L^2(T^n \to H_q^{\otimes n})$. Proposition 5.1.16 shows that for all $t > 0$, $I_{n,T}(f) = n!\, J_{n,t}(\tilde{f})$ and from Corollary 5.1.3 we are able to define $J_{n,\infty}(\tilde{f})$ as the $L^2(\Omega)$-limit of $J_{n,t}(\tilde{f})$ such that $E(J_{n,\infty}(\tilde{f}) | F_t^W) = J_{n,t}(\tilde{f})$ a.s. . Write $I_n(f) = n!\, J_{n,\infty}(\tilde{f})$ and call it the multiple Wiener integral for elements in $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$. Then using Lemma 5.5.1 and density arguments as before, I_n is also defined for elements in $L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$. Moreover, it is a linear operator from each of the above spaces to $L^2(\Omega, F^W, P)$.

The main properties of $I_n(\cdot)$ are summarized in the next result.

Lemma 5.1.3 Let $f \in L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$. Then

a) $I_n(f) = I_n(\tilde{f}) \in \mathcal{H}^{\otimes n}$ where $\mathcal{H}$ is the linear space of $(W_t)_{t \in \mathbb{R}_+}$.

b) $E(I_n(f)) = 0$.

c) If $g \in L^2(\mathbb{R}_+^m \to H_Q^{\otimes m})$ for $m \geq 1$ then

$$E(I_n(f) I_m(g)) = \delta_{nm}\; n! \int_{\mathbb{R}_+^n} \langle \tilde{f}(\underline{s}), \tilde{g}(\underline{s}) \rangle_{Q^{\otimes n}} d\underline{s}.$$

d) $$E(I_n(f))^2 = n! \int_{\mathbb{R}_+^n} \| \tilde{f}(\underline{s}) \|^2_{Q^{\otimes n}} d\underline{s} \leq n! \int_{\mathbb{R}_+^n} \| f(\underline{s}) \|^2_{Q^{\otimes n}} d\underline{s}.$$

e) For $T = [0,t]$, $t > 0$; if $I_{n,T}(f)$ is the multiple integral of f restricted to T^n then

$$E(I_n(f) \mid F_t^W) = I_{n,T}(f).$$

f) $$I_n(f) = \int_0^\infty \langle g(s), dW_s \rangle_Q$$

for $g \in M_Q$, $E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty$ and the random variable $\int_0^\infty \langle g(s), dW_s \rangle_Q$ is defined as the limit of the square integrable martingale $\int_0^t \langle g(s), dW_s \rangle_Q$.

The proof follows from the above definition, Lemma 5.1.2, Proposition 5.1.6 and Corollary 5.1.3.

5.1.2 Real valued nonlinear functionals

By a real valued nonlinear functional of $(W_t)_{t \in \mathbb{R}_+}$ we mean an element of the space $L^2(\Omega, F^W, P)$ where $F^W = F_\infty^W$. In this subsection we use the techniques developed in the last section to obtain multiple Wiener integral expansions and stochastic integral representations for elements in $L^2(\Omega, F^W, P)$. We follow the same ideas as for the one dimensional Wiener

process, as presented, for example, in Chapter VI of Kallianpur (1980).

Multiple Wiener integral orthogonal expansions Let S_I be the subspace of $L^2(\Omega) = L^2(\Omega, F^W, P)$ spanned by the multiple Wiener integrals $I_n(f_n)$ $f_n \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ $n \geq 1$ where I_n is as in Definition 5.1.5 for $q \geq r_1 + r_2$, that is

$$S_I = \overline{sp}^{L^2(\Omega)}\{I_n(f_n):\ f_n \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})\ n \geq 1\}.$$

By the definition of $I_n(\cdot)$ for elements in $L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$, we have that S_I is also equal to

$$\overline{sp}^{L^2(\Omega)}\{I_n(f_n):\ f_n \in L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})\ n \geq 1\}.$$

The next result shows the role played by the space $L^2(\mathbb{R}_+) \otimes H_Q$.

Proposition 5.1.7 Let $EXP(L^2(\mathbb{R}_+) \otimes H_Q)$ denote the Exponential Hilbert Space of $L^2(\mathbb{R}_+) \otimes H_Q$. Then

$$EXP(L^2(\mathbb{R}_+) \otimes H_Q) \overset{\eta}{\cong} S_I$$

where for $g \in EXP(L^2(\mathbb{R}_+) \otimes H_Q)$, $g = (g_0, g_1, \ldots)$ $g_n \in (L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n}$ $n \geq 0$

$$\eta(g) = \sum_{n=0}^{\infty} (n!)^{-1/2} I_n(g_n). \qquad I_0(\cdot) = 1.$$

Proof Let $g = \exp \otimes(f)$ $f \in L^2(\mathbb{R}_+) \otimes H_Q$, i.e.

$$(5.1.23)\quad \exp \otimes(f) = (f^0, f, \frac{1}{\sqrt{2!}} f^{\otimes 2}, \frac{1}{\sqrt{3!}} f^{\otimes 3}, \ldots) .$$

Then by Proposition 5.1.2 and Lemma 5.1.3 (c) and (d)

$$E(\eta(\exp \otimes(f))^2 = \sum_{n=0}^{\infty} (n!)^{-1/2} E(I_n(f^{\otimes n}))^2 = \sum_{n=0}^{\infty} (n!)^{-1} \| f^{\otimes n} \|^2_{L^2(\mathbb{R}_+^n) \otimes H_Q}$$

$$= \sum_{n=0}^{\infty} (n!)^{-1} (\| f \|^2_{L^2(\mathbb{R}_+) \otimes H_Q})^n = \| \exp \otimes(f) \|^2_{EXP(L^2(\mathbb{R}_+) \otimes H_Q)}$$

and the proof follows since elements of the form (5.1.23) span $EXP(L^2(\mathbb{R}_+)\otimes H_Q)$ (see Proposition 2.2 in Guichardet (1972)).

Q.E.D.

Proposition 5.1.8 Let $F \in L^2(\Omega, F^W, P)$. Then

$$F - E(F) = \sum_{n=1}^{\infty} I_n(f_n) \qquad \text{a.s.}$$

where $f_n \in (L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n} \simeq L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$ and $I_n(\cdot)$ $n \geq 1$ are the multiple Wiener integrals of Definition 5.1.5.

Proof From Definition 4.1.2 $H = L_1(W) \overset{I_1}{\cong} L^2(\mathbb{R}_+ \to H_Q)$ and by (4.1.26) $L^2(\Omega, F^W, P)$ can be identified with $EXP(H)$. Then

$$EXP(H) \overset{\eta_2}{\cong} EXP(L^2(\mathbb{R}_+)\otimes H_Q)$$

where

$$\eta_2(\exp \otimes(I_1(f)) = \exp \otimes(f) \qquad\qquad f \in L^2(\mathbb{R}_+)\otimes H_Q.$$

Then by Proposition 5.1.7 $L^2(\Omega, F^W, P)$ can be identified with S_I in such a way that if $F \in L^2(\Omega, F^W, P)$

$$F - E(F) = \sum_{n=1}^{\infty} I_n(f_n) \qquad \text{a.s.}$$

where $f_n \in L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$ and $I_n(\cdot)$ $n \geq 1$ are the multiple Wiener integrals of Definition 5.1.5.

Q.E.D.

The following result shows the density of orthogonal expansions of multiple Wiener integrals of elements in $L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ $q \geq r_1 + r_2$.

Corollary 5.1.4 Let $F \in L^2(\Omega, F^W, P)$, $E(F) = 0$ and $q \geq r_1 + r_2$. Then for all $\varepsilon > 0$ there exist $g_n \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ $n \geq 1$ such that

$$E(F - \sum_{n=1}^{\infty} I_n(g_n))^2 < \varepsilon .$$

Proof By Proposition 5.1.8 there exist $f_n \in L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$ $n \geq 1$ such that

$$F = \sum_{n=1}^{\infty} I_n(f_n) \qquad (L^2(\Omega)\text{-convergence}).$$

Let $q \geq r_1 + r_2$ and $\varepsilon > 0$. Then by Definition 5.1.5 and Lemma 5.1.1 for each $n \geq 1$ there exists $g_n \in L^2(\mathbb{R}_+^n \to H_q^{\otimes n})$ such that

$$E(I_n(g_n) - I_n(f_n)) < \varepsilon/2^n .$$

Then by the orthogonality of I_n for $n \neq m$

$$E(\sum_{n=1}^{\infty} I_n(g_n) - I_n(f_n))^2 = \sum_{n=1}^{\infty} E(I_n(g_n) - I_n(f_n))^2 \leq \sum_{n=1}^{\infty} \varepsilon/2^n .$$

and hence
$$E(F - \sum_{n=1}^{\infty} I_n(g_n))^2 < \varepsilon .$$

Q.E.D.

Stochastic integral representations The next result is the analog of Theorem 6.7.1 in Kallianpur (1980) for the one dimensional Wiener process.

Theorem 5.1.1 Let $F \in L^2(\Omega, F^W, P)$, $E(F) = 0$. Then

$$F(\omega) = \int_0^{\infty} \langle g(t,\omega), dW_t \rangle_Q$$

where $g \in M_Q$, $E \int_0^{\infty} \|g(t)\|_Q^2 \, dt < \infty$ and the RHS is the stochastic integral of Definition 4.2.3.

Proof Since $F \in L^2(\Omega, F^W, P)$, by Proposition 5.1.8

$$F = \sum_{n=1}^{\infty} I_n(f_n) \qquad (L^2(\Omega)\text{-convergence})$$

where $f_n \in L^2(\mathbb{R}_+^n \to H_Q^{\otimes n})$ $n \geq 1$. Then by Lemma 5.1.3 (f)

$$I_n(f_n) = \int_0^{\infty} \langle g_n(t), dW_t \rangle_Q \quad \text{a.s.}$$

where $g_n \in M_Q$ and $E \int_0^{\infty} \|g_n(t)\|_Q^2 \, dt < \infty$ $n \geq 1$. Write $g_n^* = \sum_{i=1}^{n} g_i$. Then by

linearity of the integral $\int_0^\infty \langle\cdot, dW_t\rangle_Q$

$$\sum_{i=1}^{n} I_i(f_i) = \int_0^\infty \langle g_n^*(t), dW_t\rangle_Q$$

and therefore $E(F - \int_0^\infty \langle g_n^*(t), dW_t\rangle_Q)^2 \to 0$ as $n\to\infty$. Then using Corollary 4.2.2 we obtain that

$$E\int_0^\infty \| g_n^*(t) - g_m^*(t)\|_Q^2\, dt = E[\int_0^\infty \langle g_n^*(t) - g_m^*(t), dW_t\rangle_Q] \to 0 \quad \text{as } n,m\to\infty .$$

Therefore there exists $g \in M_Q$, $E\int_0^\infty \|g(s)\|_Q^2\, ds < \infty$ such that

$$E\int_0^\infty \| g_n^*(t) - g(t)\|_Q^2\, dt \to 0 \quad \text{as } n\to\infty \quad \text{and}$$

$$E[\int_0^\infty \langle g_n^*(t), dW_t\rangle_Q - \int_0^\infty \langle g(t), dW_t\rangle_Q]^2 \to 0 .$$

Hence

$$F(\omega) = \int_0^\infty \langle h_t, dW_t\rangle_Q \qquad \text{a.s.} ,$$

Q.E.D.

The next result shows the density of the stochastic integrals $\int_0^\infty \langle\cdot, dW_t\rangle_q$ in the space of nonlinear functionals $L^2(\Omega, F^W, P)$.

Corollary 5.1.5 Let $F \in L^2(\Omega, F^W, P)$, $E(F) = 0$. Then for all $q \geq r_1 + r_2$ and $\varepsilon > 0$ there exists $g \in M_q$, $E\int_0^\infty \|g(s)\|_q^2\, ds < \infty$ such that

$$E(F - \int_0^\infty \langle g_t, dW_t\rangle_q)^2 < \varepsilon .$$

The proof follows from the last theorem and Definition 4.2.3.

Representation of real valued square integrable martingales

Theorem 5.1.2 Let $(M_t, F_t^W)_{t\in\mathbb{R}_+}$ be a square integrable martingale, with $M_0 = 0$. Then (M_t) has a continuous modification, say $(\tilde{M}_t)$, which is given by the stochastic integral

$$\tilde{M}_t(\omega) = \int_0^t \langle g(s,\omega), dW_s \rangle_Q \qquad \text{a.s.}$$

for every $t > 0$, where $g \in M_Q$, i.e. $g(s,\omega)$ is jointly measurable, $g(s,\cdot)$ is F_s^W-adapted and

$$E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty .$$

Proof Since $(M_t)_{t \in \mathbb{R}_+}$ is a square integrable martingale, i.e. $\sup_{0 \le t < \infty} E(M_t^2) < \infty$, then M_∞ exists as the mean square limit (and also as the almost sure limit) of M_t $t \to \infty$, and moreover, M_∞ is F_∞^W-measurable and $EM_\infty^2 < \infty$, i.e. $M_\infty \in L^2(\Omega, F, P)$. Then by Theorem 5.1.1

$$M_\infty = \int_0^\infty \langle g(t), dW_t \rangle_Q \qquad \text{a.s.}$$

where $g \in M_Q$, $E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty$. Then by Corollary 4.2.2

$$M_t = E(M_\infty | F_t^W) = \int_0^t \langle g(t), dW_t \rangle_Q \quad \text{a.s.}$$

and the stochastic integral has a continuous version which is the required modification of M_t.

Q.E.D.

5.2 Φ'-valued multiple Wiener integrals

Let Φ and $\Phi^{\otimes n}$ $n \ge 1$, be as in Section 4.1.1 and denote by $L((\Phi^{\otimes n})', \Phi')$ the class of continuous linear operators from $(\Phi^{\otimes n})'$ to Φ'. In this section we define multiple Wiener integrals of the form

$$Y_{n,T}(f) = \int_T \overset{n}{\cdots} \int f(\underline{t})\, dW_{t_1} \ldots dW_{t_n} \qquad \underline{t} = (t_1, \ldots, t_n)$$

where $f(\underline{t})$ is a non-random element in $L((\Phi^{\otimes n})', \Phi')$ and $T = [0, T_0]$ for all $n \ge 1$ and $T_0 > 0$ (Section 5.2.1). Then we construct multiple stochastic integral expansions and stochastic integral representations for Φ'-valued

nonlinear functionals and Φ'-valued square integrable martingales of $(W_t)_{t\in\mathbb{R}_+}$ (Section 5.2.2). A Wiener type decomposition of the space of Φ'-valued nonlinear functionals is obtained in Theorem 5.2.2 as the inductive limit of the spaces $L^2(\Omega\to H_{-r})$ $r\geq 0$.

5.2.1 Multiple Wiener integrals for $L((\Phi^{\otimes n})',\Phi')$-functions

Throughout this section $n\geq 1$ is fixed but arbitrary.

Definition 5.2.1 A measurable function $f:\mathbb{R}_+^n\to L((\Phi^{\otimes n})',\Phi')$ is said to belong to the class $\Theta_Q((\Phi^{\otimes n})',\Phi')$ if for all $T=[0,T_o]$, $T_o>0$

$$(5.2.1)\qquad \int_{T^n} Q^{\otimes n}(f^*_{\underline{t}}(\phi),f^*_{\underline{t}}(\phi))\,d\underline{t}<\infty \qquad \forall\ \phi\in\Phi$$

where $f^*_{\underline{t}}:\Phi\to\Phi^{\otimes n}$ is the adjoint of $f_{\underline{t}}$ and $\underline{t}=(t_1,\dots,t_n)$.

Theorem 5.2.1 Let $f\in\Theta_Q((\Phi^{\otimes n})',\Phi')$. Then for each $T_o>0$ there exists a Φ'-valued element $Y_{n,T}(f)$, $T=[0,T_o]$ such that

$$(5.2.2)\qquad Y_{n,T}(f)[\phi] = I_{n,T}(f^*(\phi)) \quad \text{a.s.} \quad \forall\ \phi\in\Phi$$

where $I_{n,T}$ is the real valued multiple Wiener integral of Definition 5.1.3 for elements in $L^2(T^n\to H_Q^{\otimes n})$. $Y_{n,T}$ is called the n^{th} Φ'-valued multiple Wiener integral.

Proof We prove this theorem in a very similar way to Proposition 4.2.3. First note that for each $\phi\in\Phi$ $I_{n,T}(f^*(\phi))$ is well defined since from (5.2.1) $f^*(\phi)\in L^2(T^n\to H_Q^{\otimes n})$.

Next for $\phi\in\Phi$ define $V_f(\phi)=V_{f,T}(\phi)$ as

$$(5.2.3)\qquad V_f^2(\phi) = \int_{T^n} Q^{\otimes n}(f^*_{\underline{t}}(\phi),f^*_{\underline{t}}(\phi))\,d\underline{t}.$$

Since $Q^{\otimes n}$ is $\Phi^{\otimes n}$-continuous, using Fatou's lemma one can show (as in

Proposition 4.2.3) that V_f is a lower semicontinuous function on Φ. Moreover, V_f is a non-negative function on Φ that satisfies conditions (a), (b) and (c) of Lemma 4.1.1 (use triangle inequality to prove (a) and (5.2.1) to show (c)). Then this lemma implies that V_f is a continuous function on Φ and therefore there exist $\theta_f = \theta_{f,T}$ and $r_f = r_{f,T}$ such that

(5.2.4) $$V_f^2(\phi) \leq \theta_f \|\phi\|_{r_f}^2 \qquad \forall\ \phi \in \Phi .$$

Next let $\{\phi_j\}_{j\geq 1}$ and $\{\lambda_j\}_{j\geq 1}$ be as in Section 4.1.1, $q_f = q_{f,T}$ such that $q_f \geq r_f + r_1$ and write $\tilde{\phi}_j = (1+\lambda_j)^{-q_f}\phi_j$ $j \geq 1$. Then $\{\tilde{\phi}_j\}_{j\geq 1}$ is a CONS for H_{q_f} and we denote by $\{\psi_j\}_{j\geq 1}$ the CONS for H_{-q_f} dual to $\{\tilde{\phi}_j\}_{j\geq 1}$, i.e. $\langle\psi_k, \tilde{\phi}_j\rangle_{-q_f} = \delta_{k_j}$.

Define $Y_{n,T}(f)[\tilde{\phi}_j] = I_{n,T}(f^*(\tilde{\phi}_j))$ $j \geq 1$. Then by Definition 5.1.3 and Proposition 5.1.4 (e)

$$\begin{aligned}
&\sum_{j=1}^{\infty} E(Y_{n,T}(f)[\tilde{\phi}_j])^2 \\
&\leq n! \sum_{j=1}^{\infty} \int_{T^n} Q^{\otimes n}(f^*_{\underline{t}}(\tilde{\phi}_j), f^*_{\underline{t}}(\tilde{\phi}_j))d\underline{t} = n! \sum_{j=1}^{\infty} V_f^2(\tilde{\phi}_j) \\
&\leq n!\ \theta_f \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|_{r_f}^2 = n!\ \theta_f \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(q_f - r_f)} \leq n\ \theta_f \theta_1 < \infty
\end{aligned}$$

where θ_1 is as in (4.1.4). Then $\sum_{j=1}^{\infty}(Y_{n,T}(f)[\tilde{\phi}_j])^2 < \infty$ a.s. . Let $\Omega_1 = \{\omega: \sum_{j=1}^{\infty}(Y_{n,T}(f)[\tilde{\phi}_j](\omega))^2 < \infty\}$, then $P(\Omega_1) = 1$. Define

(5.2.5) $$\tilde{Y}_{n,T}(f)(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j](\omega)\ \psi_j & \omega \in \Omega_1 \\ 0 & \omega \notin \Omega_1 . \end{cases}$$

Then $\tilde{Y}_{n,T}(f) \in H_{-q_f}$ a.s. for $q_f \geq r_f + r_1$ and then $\tilde{Y}_{n,T}(f) \in \Phi'$ a.s. . From now on write $Y_{n,T}(f) = \tilde{Y}_{n,T}(f)$.

Next if $\phi \in \Phi$, then $\phi \in H_{q_f}$ $q_f \geq r_f + r_1$ and

$$Y_{n,T}(f)[\phi] = \sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j]\psi_j[\phi] = \sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j] \langle\phi,\tilde{\phi}_j\rangle_{q_f}$$

$$= \sum_{j=1}^{\infty} Y_{n,T}(f)[\langle\phi,\tilde{\phi}_j\rangle_{q_f}\tilde{\phi}_j] .$$

Thus for all $\phi \in \Phi$, since $Y_{n,T}(f)[\tilde{\phi}_j] = I_{n,T}(f^*(\tilde{\phi}_j))$

$$(5.2.6) \qquad Y_{n,T}(f)[\phi] = \lim_{m\to\infty} I_{n,T}(f^*(\sum_{j=1}^{m} \langle\phi,\tilde{\phi}_j\rangle_{q_f}\tilde{\phi}_j)) \quad \text{a.s. .}$$

On the other hand, since $\sum_{j=1}^{m}\langle\phi,\tilde{\phi}_j\rangle_{q_f} \tilde{\phi} \underset{m\to\infty}{\to} \phi$ on H_{q_f} then $V_f(\sum_{j=1}^{m}\langle\phi,\tilde{\phi}_j\rangle_{q_f} \phi_j - \phi) \underset{m\to\infty}{\to} 0$ which implies using (5.2.3) and Proposition 5.1.4 (e) that

$$E(I_{n,T}(f^*(\phi)) - I_{n,T}(f^*(\sum_{j=1}^{m} \langle\phi,\tilde{\phi}_j\rangle_{q_f} \tilde{\phi}_j)))^2 \to 0 \quad \text{as } m\to\infty .$$

Then from (5.2.6) for all $T = [0,T_o]$, $T_o > 0$

$$Y_{n,T}(f)[\phi] = I_{n,T}(f^*(\phi)) \quad \text{a.s.} \quad \forall\, \phi \in \Phi .$$

Q.E.D.

The following properties of the Φ'-valued multiple Wiener integral $Y_{n,T}(f)$ follow from the last theorem, (5.2.2), Proposition 5.1.4, Definition 5.1.3 and Corollary 5.1.1.

<u>Proposition 5.2.1</u> Let $f,g \in \Theta_Q((\Phi^{\otimes n})',\Phi')$. Then for each $T = [0,T_o]$ $T_o > 0$

a) $Y_{n,T}(af+bg) = aY_{n,T}(f) + bY_{n,T}(g)$ a.s. $a,b \in \mathbb{R}$.

b) $E(Y_{n,T}(f)[\phi]) = 0$ $\forall\, \phi \in \Phi$.

c) If $g \in \Theta_Q((\Phi^{\otimes m})',\Phi')$

$$E(Y_{n,T}(f)[\phi]\; Y_{m,T}(g)[\psi])$$

$$= \delta_{nm}\, n! \int_{T^n} Q^{\otimes n}(\widetilde{f^*_{\underline{t}}(\phi)}, \widetilde{g^*_{\underline{t}}(\phi)})d\underline{t} \qquad \forall\ \phi,\ \psi \in \Phi$$

where $\widetilde{f^*_t(\phi)}$ is the symmetrization (Corollary 5.1.1) of $f^*_t(\phi)$ on $\Phi^{\otimes n}$.

d) $E(Y_{n,T}(f)[\phi])^2 \leq n! \int_{T^n} Q^{\otimes n}(f^*_{\underline{t}}(\phi), f^*_{\underline{t}}(\phi))d\underline{t}$.

Proposition 5.2.2 Let $n \geq 1$ and $g: \mathbb{R}_+ \to L((\Phi^{\otimes n})', \Phi')$. Define the symmetrization $\tilde{g}$ of g such that for each $t \in \mathbb{R}_+$ and $\phi \in \Phi$ $\tilde{g}^*_t(\phi) = \widetilde{g^*_t(\phi)}$ where $\widetilde{g^*_t(\phi)}$ is the symmetrization of $g^*_t(\phi)$ on $\Phi^{\otimes n}$ of Corollary 5.1.1. If $f \in \Theta_Q((\Phi^{\otimes n})', \Phi')$ then

a) $\tilde{f} \in \Theta_Q((\Phi^{\otimes n})', \Phi')$ and for each $T = [0, T_o]$, $T_o > 0$

$$Y_{n,T}(\tilde{f}) = Y_{n,T}(f) \quad \text{a.s. .}$$

b) For each $T = [0, T_o]$, $T_o > 0$ there exists $q_{f,T} > 0$ such that a.e. $\underline{t} \in T^n$ $f_{\underline{t}}$ and $\tilde{f}_{\underline{t}}$ are Hilbert-Schmidt operators from $H_Q^{\otimes n}$ to $H_{-q_{f,T}}$ and

$$\text{(5.2.7)} \qquad E\|Y_{n,T}(f)\|^2_{-q_{f,T}} = n!\ \|\tilde{f}\|^2_{L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-q_{f,T}}))}$$

$$\leq n!\ \|f\|^2_{L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-q_{f,T}}))} < \infty$$

where $\sigma_2(H_Q^{\otimes n}, H_{-q_{f,T}})$ denotes the Hilbert space of Hilbert-Schmidt operators from $H_Q^{\otimes n}$ to $H_{-q_{f,T}}$.

Proof a) By Corollary 5.1.1 for each $t \in \mathbb{R}_+$ and $\phi \in \Phi$ $\tilde{f}^*_t(\phi) = \widetilde{f^*_t(\phi)} \in \Phi^{\otimes n}$ and by Corollary 5.1.2 for each $T = [0, T_o]$, $T_o > 0$ and $\phi \in \Phi$

$$\int_{T^n} Q^{\otimes n}(\widetilde{f^*_{\underline{t}}(\phi)}, \widetilde{f^*_{\underline{t}}(\phi)})d\underline{t} \leq \int_{T^n} Q^{\otimes n}(f^*_{\underline{t}}(\phi), f^*_{\underline{t}}(\phi))d\underline{t}$$

which is finite for each T since $f \in \Theta_Q((\Phi^{\otimes n})', \Phi')$, proving that

$\tilde{f} \in \Theta_Q((\Phi^{\otimes n})', \Phi')$.

Thus $Y_{n,T}(\tilde{f})$ is defined for each $T = [0,T_o]$ $T_o > 0$ and from Theorem 5.2.1, Definition 5.1.3, Proposition 5.1.4 (a) and (5.2.2) we have that

$$Y_{n,T}(\tilde{f}) = Y_{n,T}(f) \quad \text{a.s.} \quad \text{for each } T = [0,T_o], T_o > 0.$$

b) For $T = [0,T_o]$, $T_o > 0$ let $V_{f,T}$ and $q_{f,T}$ be as in the proof of Theorem 5.2.1 and take $\{(1+\lambda_j)^{q_{f,T}}\phi_j\}_{j\geq 1}$ a CONS in $H_{-q_{f,T}}$. Then $Y_{n,T}(f) \in H_{-q_{f,T}}$ a.s. and

$$E\|Y_{n,T}(f)\|^2_{-q_{f,T}} = E\sum_{j=1}^{\infty} \langle Y_{n,T}(f), (1+\lambda_j)^{q_{f,T}}\phi_j\rangle^2_{-q_{f,T}}$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{2q_{f,T}} E\langle Y_{n,T}(f), \phi_j\rangle^2_{-q_{f,T}}$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q_{f,T}} E(Y_{n,T}(f)[\phi_j])^2 \qquad \text{(by (4.1.8))}$$

$$= n! \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q_{f,T}} \int_{T^n} \|f^*_{\underline{t}}(\phi_j)\|^2_{Q^{\otimes n}} d\underline{t} \qquad \text{(Proposition 5.2.1 (d))}$$

$$= n! \sum_{j=1}^{\infty} \int_{T^n} \|\tilde{f}^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}}\phi_j)\|^2_{Q^{\otimes n}} d\underline{t} = n! \int_{T^n} \Big(\sum_{j=1}^{\infty} \|\tilde{f}^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}}\phi_j\|^2_{Q^{\otimes n}}\Big) d\underline{t}$$

$$\leq n! \int_{T^n} \Big(\sum_{j=1}^{\infty} \|f^*_{\underline{t}}(1+\lambda_j)^{-q_{f,T}}\phi_j)\|^2_{Q^{\otimes n}} d\underline{t} \qquad \text{(Corollary 5.1.2)}$$

$$= n! \int_{T^n} \sum_{j=1}^{\infty} Q^{\otimes n}(f^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}}\phi_j), f^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}}\phi_j)) d\underline{t}$$

$$= n! \sum_{j=1}^{\infty} V^2_{f,T}((1+\lambda_j)^{-q_{f,T}}\phi_j) \qquad \text{(by 5.2.4)}$$

$$\leq n!\, \theta_{f,T} \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} = n!\, \theta_{f,T}\theta_1 < \infty .$$

Then a.e. $\underline{t} \in T^n$

$$\| f^*_{\underline{t}} \|^2_{\sigma_2(H_{q_{f,T}}, H_Q^{\otimes n})} = \sum_{j=1}^{\infty} \| f^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}} \phi_j) \|^2_{Q^{\otimes n}} < \infty$$

and

$$\| \tilde{f}^*_{\underline{t}} \|^2_{\sigma_2(H_{q_{f,T}}, H_Q^{\otimes n})} = \sum_{j=1}^{\infty} \| \tilde{f}^*_{\underline{t}}((1+\lambda_j)^{-q_{f,T}} \phi_j) \|^2_{Q^{\otimes n}} < \infty$$

where $\{(1+\lambda_j)^{-q_{f,T}}\phi_j\}_{j\geq 1}$ is a CONS for $H_{q_{f,T}}$. Then (5.2.7) follows since

$$\| f_{\underline{t}} \|^2_{\sigma_2(H_Q^{\otimes n}, H_{-q_{f,T}})} = \| f^*_{\underline{t}} \|^2_{\sigma_2(H_{q_{f,T}}, H_Q^{\otimes n})} .$$

Q.E.D.

We now extend the definition of Y_n to functions $f: \mathbb{R}^n_+ \to L((\Phi^{\otimes n})', \Phi')$. Proposition 5.2.2 (b) suggests that it is enough to construct multiple Wiener integrals for functions $f: \mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s})$ for $s \geq r_1 + r_2$, as we now do.

<u>Proposition 5.2.3</u> *Let $s \geq q_1 + q_2$ and $n \geq 1$ be fixed but arbitrary.* Let $f \in L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$. Then there exists an H_{-s}-valued element $Y_n(f)$ called the multiple Wiener integral for functions in $L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$ such that

(5.2.8) $$Y_n(f)[\phi] = I_n(f^*(\phi)) \quad \text{a.s.} \quad \forall\ \phi \in H_s$$

where I_n is the multiple Wiener integral of Definition 5.1.5 for elements in $L^2(\mathbb{R}^n_+ \to H_Q^{\otimes n})$. Moreover, $Y_n(f)$ satisfies the following properties

a) If $g \in L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$ and $a, b \in \mathbb{R}$.

$$Y_n(af+bg) = aY_n(f) + bY_n(g) \qquad \text{a.s. .}$$

b) If $g \in L^2(\mathbb{R}^m_+ \to \sigma_2(H_Q^{\otimes m}, H_{-s}))$ then

$$E\langle Y_n(f), Y_m(g)\rangle_{-s} = \delta_{n,m} n!\ \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))}$$

and

$$E\| Y_n(f) \|^2_{-s} = n! \ \|\tilde{f}\|_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))}$$

$$\leq n! \ \| f \|_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))} < \infty .$$

c) For $T = [0,t]$, $t > 0$ if $I_{n,T}$ is the real valued multiple Wiener integral for elements in $L^2(T^n \to H_Q^{\otimes n})$ then

$$E(Y_n(f)[\phi] \mid F_t^W) = I_{n,T}(f^*(\phi)) \text{ a.s. } \forall \ \phi \in H_s .$$

d) For each $\phi \in H_s$ there exists $g_\phi \in M_Q$, $E \int_0^\infty \| g_\phi(s) \|_Q^2 ds < \infty$ such that

$$Y_n(f)[\phi] = \int_0^\infty \langle g_\phi(s), dW_s \rangle_Q \quad \text{a.s.}$$

where the RHS is the real valued stochastic integral of Definition 4.2.3.

<u>Proof</u> The first part is proved as in Theorem 5.2.1 using Definition 5.1.5 for elements in $L^2(\mathbb{R}^n_+ \to H_Q^{\otimes n})$. (a) follows from (5.2.8) and the linearity property on $I_n(\cdot)$. (c) and (d) follow from (5.2.8) and Lemma 5.1.3 (e) and (d). The second part of (b) follows as in the proof of (b) in Proposition 5.2.2. To prove the first part of (b) let $\{e_k = (1+\lambda_k)^s \phi_k\}_{k\geq 1}$ be a CONS for H_{-s}, then using (5.2.8) and Lemma 5.1.3 (c)

$$E\langle Y_n(f), Y_m(g)\rangle_{-s} = \sum_{k=1}^\infty (1+\lambda_k)^{-2s} E(I_n(f^*(\phi_k)) I_m(g^*(\phi_k)))$$

$$= \delta_{nm} n! \int_{\mathbb{R}^n_+} \sum_{k=1}^\infty \langle \widetilde{f^*_{\underline{t}}(e_k)}, \widetilde{g^*_{\underline{t}}(e_k)} \rangle_{Q^{\otimes n}} d\underline{t}$$

$$= \delta_{nm} \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-s}))} .$$

Q.E.D.

We will see in the next section that the multiple Wiener integrals $Y_n(\cdot)$ for elements in $L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$ $s \geq r_1 + r_2$ form a complete system in the space of Φ'-valued nonlinear functionals of $(W_t)_{t \in \mathbb{R}_+}$.

The next result will be useful for the representation of Φ'-valued nonlinear functionals. It relates the Φ'-multiple Wiener integral Y_n with the Φ'-stochastic integral of Proposition 4.2.3 and it is an infinite dimensional analog of Lemma 6.7.2 in Kallianpur (1980).

Proposition 5.2.4 Let $n \geq 1$, $s \geq q_1 + q_2$ and $f \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$. Then there exists a non-anticipative $\sigma_2(H_Q, H_{-s})$-valued process $h(t,\omega)$ such that

$$E \int_0^\infty \| h(t,\omega) \|^2_{\sigma_2(H_Q, H_{-s})} dt < \infty$$

and

$$Y_n(f) = \int_0^\infty h(t,\omega) dW_t$$

where the RHS of the last expression is the Φ'-valued stochastic integral of Proposition 4.2.5.

Proof By Proposition 5.2.3 (d) for each $\phi \in H_s$

$$(5.2.9) \qquad Y_n(f)[\phi] = \int_0^\infty \langle g_\phi(t), dW_t \rangle_Q = I_n(f^*(\phi)) \quad \text{a.s.}$$

where $g_\phi \in M_Q$, $E(\int_0^\infty \| g_\phi(s) \|_Q^2 ds) < \infty$. Let $\{e_k\}_{k\geq 1}$ be a CONS for H_s and define $h^*(t,\omega)(e_k) = g_{e_k}(t,\omega)$ $k \geq 1$. Then $h^*(t)(e_k)$ is H_Q-valued and $h^*(t)(e_k) \in M_Q$ $k \geq 1$. Next

$$\int_0^\infty E(\sum_{k=1}^\infty \| h^*(t)(e_k) \|_Q^2) dt = \sum_{k=1}^\infty \int_0^\infty E \| g_{e_k}(t) \|_Q^2 dt$$

$$= \sum_{k=1}^\infty E(\int_0^\infty \langle g_{e_k}(t), dW_t \rangle_Q)^2 \qquad \text{(by Corollary 4.2.1)}$$

$$= \sum_{k=1}^{\infty} E(Y_n(f)[e_k])^2 \qquad \text{(by (5.2.9))}$$

$$\leq n! \sum_{k=1}^{\infty} \| f^*(e_k) \|^2_{L^2(\mathbb{R}^n_+ \to H_Q^{\otimes n})} \qquad \text{(by Lemma 5.1.3 (d))}$$

$$= n! \| f^* \|^2_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_s,H_Q))} = n! \| f \|^2_{L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q,H_{-s}))} < \infty .$$

Then

$$h^*(t)(\cdot) = \sum_{k=1}^{\infty} \langle \cdot, e_k \rangle_s \; h^*(t)(e_k)$$

defines an a.s. dtdP linear operator from H_s to H_Q. Moreover, from the above calculations $h^*(t,\omega) \in \sigma_2(H_Q,H_s)$ a.s. dtdP. Then

$$\int_0^{\infty} E\| h(t) \|^2_{\sigma_2(H_Q,H_{-s})} dt = \int_0^{\infty} E\| h^*(t) \|^2_{\sigma_2(H_s,H_Q)} dt < \infty$$

and the proposition follows by the definition of the Φ'-valued stochastic integral $\int_0^{\infty} h(t,\omega) dW_t$ of Proposition 4.2.5.

Q.E.D.

5.2.2 Φ'-valued nonlinear functionals

Let $F^W = F^W_{\infty}$. By a Φ'-valued nonlinear functional of $(W_t)_{t\in\mathbb{R}_+}$ we mean a Φ'-valued random element $F: \Omega \to \Phi'$ such that F is $F^W \to \mathcal{B}(\Phi')$ measurable, and

$$E(F[\phi])^2 < \infty \qquad \forall \; \phi \in \Phi .$$

We denote by $L^2(\Omega \to \Phi') = L^2((\Omega,F^W,P) \to \Phi')$ the linear space of all Φ'-valued nonlinear functionals of $(W_t)_{t\in\mathbb{R}_+}$. Observe that it is not a Hilbert space.

For $r \geq 0$ let $L^2(\Omega \to H_{-r}) = L^2((\Omega,F^W,P) \to H_{-r})$ be the Hilbert space of all F^W-measurable elements $F: \Omega \to H_{-r}$ such that $E(\| F \|^2_{-r}) < \infty$. The Hilbert space $L^2(\Omega \to H_{-r})$ is called the space of H_{-r}-valued nonlinear

functionals of $(W_t)_{t\in\mathbb{R}_+}$.

Wiener decomposition of the space $L^2(\Omega\to\Phi')$ Let $H = L_1(W)$ be the Gaussian space of $(W_t)_{t\in\mathbb{R}_+}$ defined in (4.1.25) and $H^{\circledS n}$ be its n-fold symmetric tensor product. For fixed $s > 0$ and $n \geq 1$ let

$$G_n(H_{-s}) = \{\eta\in L^2(\Omega\to H_{-s}) : \eta[\phi]\in H^{\circledS n} \quad \forall\ \phi\in H_s\} . \tag{5.2.10}$$

Recall that for all $n\geq 1$ $H^{\circledS n}$ is a subspace of $L^2(\Omega, F^W, P)$.

Theorem 5.2.2 (Wiener decomposition of $L^2(\Omega\to\Phi')$). The linear space $L^2(\Omega\to\Phi')$ is a complete locally convex space in the topology given by the strict inductive limit of the Hilbert spaces $L^2(\Omega\to H_{-r})$ $r\geq 0$ and

$$L^2(\Omega\to\Phi') = \varinjlim_{r\to\infty} \Big(\sum_{n\geq 0} \oplus\, G_n(H_{-r})\Big). \tag{5.2.11}$$

The proof of this theorem is based on the following lemmas.

Lemma 5.2.1

$$L^2(\Omega\to\Phi') = \bigcup_{r=0}^{\infty} L^2(\Omega\to H_{-r}). \tag{5.2.12}$$

Proof Let $F\in L^2(\Omega\to H_{-r})$ $r\geq 0$. Then $F[\phi]$ is F^W-measurable for all $\phi\in\Phi$ and $E(F[\phi])^2 \leq \|\phi\|_r^2\, E\|F\|_{-r}^2 < \infty$, i.e. $F\in L^2(\Omega\to\Phi')$ and hence

$$\bigcup_{r=0}^{\infty} L^2(\Omega\to H_{-r}) \subset L^2(\Omega\to\Phi') . \tag{5.2.13}$$

Next let $F\in L^2(\Omega\to\Phi')$ and for all $\phi\in\Phi$ define $V^2(\phi) = E(F[\phi])^2$. Then

$$V^2(\phi) < \infty \qquad \forall\ \phi\in\Phi . \tag{5.2.14}$$

As in the proof of Proposition 4.2.3 and Theorem 5.2.1, using the continuity of F on Φ and Fatou's lemma, one can show that $V(\phi)$ is a lower

semicontinuous function of Φ. Moreover it is non-negative and satisfies conditions (a), (b) and (c) of Lemma 4.1.1. Then $V(\phi)$ is a continuous function on Φ and hence there exist $\theta_F > 0$ and $r_F > 0$ such that

$$(5.2.15) \qquad V^2(\phi) = E(F(\phi))^2 \le \theta_F \|\phi\|^2_{r_F} \quad \forall\ \phi \in \Phi .$$

Let $r \ge r_F + r_1$, then the imbedding of H_r into H_{r_F} is a Hilbert-Schmidt map. Take $\tilde{\phi}_j = (1+\lambda_j)^{-r}\phi_j$, then $\{\tilde{\phi}_j\}_{j\ge1}$ is a CONS in H_r and

$$E\left(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2\right) = \sum_{j=1}^{\infty} E(F[\tilde{\phi}_j])^2 = \sum_{j=1}^{\infty} V^2(\tilde{\phi}_j)$$

$$\le \theta_F \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|^2_{r_F} = \theta_F \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(r-r_F)} \le \theta_F \theta_1 < \infty$$

where θ_1 is as in (4.1.4). Then $\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2 < \infty$ a.s., and if $\{\psi_j\}_{j\ge1}$ is the CONS in H_{-r} dual to $\{\tilde{\phi}_j\}_{j\ge1}$

$$P\left(\tilde{F}(\omega) = \sum_{j=1}^{\infty} F[\tilde{\phi}_j](\omega)\psi_j < \infty\right) = 1$$

and $\tilde{F} \in H_{-r}$ a.s. . Moreover,

$$E\|\tilde{F}\|^2_{-r} = \sum_{j=1}^{\infty} E<\tilde{F},(1+\lambda_j)^r\phi_j>^2_{-r} = E\left(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2\right) < \infty .$$

It remains to show that for each $\phi \in \Phi$ $F[\phi] = \tilde{F}[\phi]$. By using (5.2.15) since $\sum_{j=1}^{m} <\phi,\phi_j>_r \tilde{\phi}_j \underset{m\to\infty}{\to} \phi$ in H_r

$$E\left(F[\phi] - \sum_{j=1}^{m} F[\tilde{\phi}_j]\psi_j[\phi]\right)^2 = E\left(F[\phi - \sum_{j=1}^{m} <\phi,\tilde{\phi}_j>_r \tilde{\phi}_j]\right)^2$$

$$\le \theta_F \|\phi - \sum_{j=1}^{m} <\phi,\tilde{\phi}_j>_r \tilde{\phi}_j\|^2_r \to 0 \quad \text{as } m\to\infty$$

and therefore for each $\phi \in \Phi$ $F[\phi] = \tilde{F}[\phi]$ a.s. .

Thus if $F \in L^2(\Omega \to \Phi')$ there exists $r \ge 0$ such that $F \in L^2(\Omega \to H_{-r})$ which together with (5.2.13) implies (5.2.12).

Q.E.D.

<u>Lemma 5.2.2</u> For $r \geq 0$ and $n \geq 1$ let $G_n(H_{-r})$ be as in (5.2.10). Then for fixed $r \geq 0$ $G_n(H_{-r})$ $n \geq 1$ are Hilbert subspaces of $L^2(\Omega \to H_{-r})$ such that if $n \neq m$ $G_n(H_{-r})$ and $G_m(H_{-r})$ are orthogonal in $L^2(\Omega \to H_{-r})$.

<u>Proof</u> We first prove that for each $n \geq 1$ $G_n(H_{-r})$ is a Hilbert subspace of $L^2(\Omega \to H_{-r})$. Let $\{\eta_k\}_{k\geq 1}$ be a Cauchy sequence in $G_n(H_{-r})$, then $\eta_k \underset{k\to\infty}{\to} \eta$ in $L^2(\Omega \to H_{-r})$. Since for each $\phi \in H_{-r}$

$$|\eta_k[\phi] - \eta[\phi]| \leq ||\phi||_r ||\eta_k - \eta||_{-r}$$

then for each $\phi \in H_r$ $\eta_k[\phi] \to \eta[\phi]$ in $L^2(\Omega, F^W, P)$.

But by hypothesis $\eta_k[\phi] \in H^{\circledcirc n}$. Then since $H^{\circledcirc n}$ is a closed subspace of $L^2(\Omega, F^W, P)$, $\eta[\phi] \in H^{\circledcirc n}$ for all $\phi \in H_{-r}$ i.e. $\eta \in G_n(H_{-r})$, proving that $G_n(H_{-r})$ is a closed subspace of $L^2(\Omega \to H_{-r})$ for $n \geq 1$.

Next if $n \neq m$, let $\eta_n \in G_n(H_{-r})$, $\eta_m \in G_m(H_{-r})$ and $\{e_k\}_{k\geq 1} = \{(1+\lambda_k)^r \phi_k\}_{k\geq 1}$ a CONS for H_{-r}. Then

$$E\langle\eta_n, \eta_m\rangle_{-r} = \sum_{j=1}^{\infty} E(\langle\eta_m, e_k\rangle_{-r} \langle\eta_n, e_k\rangle_{-r}) = \sum_{k=1}^{\infty} (1+\lambda_k)^{-2r} E(\eta_n[\phi_k]\eta_m[\phi_k]) = 0$$

since $\eta_n[\phi_k] \in H^{\circledcirc n}, \eta_m[\phi_k] \in H^{\circledcirc m}$ all $k \geq 1$ and $H^{\circledcirc n}$ and $H^{\circledcirc m}$ are orthogonal in $L^2(\Omega, F^W, P)$.

Q.E.D.

The following result is the Wiener decomposition of the space $L^2(\Omega \to H_{-r})$. A proof of it appears in Miyahara (1981) for a general Hilbert space K, i.e. for $L^2(\Omega \to K)$.

<u>Lemma 5.2.3</u> For each $r \geq 0$

$$L^2(\Omega \to H_{-r}) = \sum_{n\geq 0} \oplus G_n(H_{-r}).$$

<u>Proof</u> Let $\{e_k\}_{k\geq 1}$ be a CONS for H_{-r} and $\eta \in L^2(\Omega \to H_{-r})$, i.e. η is F^W-measurable and $E\|\eta\|^2_{-r} < \infty$. Then

$$\eta(\omega) = \sum_{k=1}^{\infty} \langle \eta(\omega), e_k \rangle_{-r} e_k \qquad (L^2(\Omega \to H_{-r}) \text{ convergence})$$

where $E(\langle \eta, e_k \rangle^2_{-r}) < \infty$ all $k \geq 1$, i.e. $\langle \eta, e_k \rangle_{-r} \in L^2(\Omega, F^W, P)$ and therefore

$$\langle \eta, e_k \rangle^2_{-r} = \sum_{n=1}^{\infty} x_n^k \qquad (L^2(\Omega)\text{-convergence})$$

where for each k $x_n^k \in H^{\otimes n}$ $n \geq 1$.

Define

$$\eta_n = \sum_{k=1}^{\infty} x_n^k e_k .$$

We now prove that for each $n \geq 1$ $\eta_n \in G_n(H_{-r})$. Note that

$$E\|\sum_{k=m}^{\ell} x_n^k e_k\|^2_{-r} = E(\sum_{k=m}^{\ell} |x_n^k|^2) = \sum_{k=m}^{\ell} E|x_n^k|^2$$

$$\leq \sum_{k=m}^{\ell} E(\langle \eta, e_k \rangle^2_{-r}) \to 0 \quad \text{as } m, \ell \to \infty .$$

Then $\{\sum_{k=1}^{m} x_n^k e_k\}_{k\geq 1}$ is a Cauchy sequence in $L^2(\Omega \to H_{-r})$ and therefore converges to a limit denoted by η_n.

Next if $\phi \in H_r$ $\eta_n[\phi] = \sum_{k=1}^{\infty} x_n^k e_k[\phi] \in H^{\otimes n}$ $n \geq 1$, i.e. $\eta_n \in G_n(H_{-r})$ $n \geq 1$.

By construction of η_m $m \geq 1$ if $\sum_{m=1}^{\infty} \eta_m$ converges in $L^2(\Omega \to H_{-r})$ it must converge to η. Thus it remains to prove that $\sum_{m=1}^{n} \eta_m$ is a Cauchy sequence:

$$E\|\sum_{m=n}^{\ell} \eta_m\|^2_{-r} = \sum_{m=n}^{\ell} E\|\eta_m\|^2_{-r} = \sum_{k=1}^{\infty} \sum_{m=n}^{\infty} E|x_m^k|^2 .$$

But for each k $\sum_{m=n}^{\ell} E|x_m^k|^2 \to 0$ as $n, \ell \to \infty$, therefore

$$E\|\sum_{m=n}^{\infty} \eta_m\|^2_{-e} \to 0 \quad \text{as } n, \ell \to \infty .$$

Then $\sum_{n=1}^{\infty} \eta_n$ is an element of $L^2(\Omega \to H_{-r})$ and of $\sum_{n\geq 0} \oplus\, G_n(H_{-r})$, which is equal to η a.e..

Proof of Theorem 5.2.2 Since $L^2(\Omega \to H_{-r}) \subset L^2(\Omega \to H_{-(r+1)})$ for all $r \geq 0$ the theorem follows by Lemmas 5.2.1 and 5.2.3 and the following result (Theorem V.15 of Reed and Simon (1980)): Let X be a real vector space and X_n be a family of subspaces with $X_n \subset X_{n+1}$, $X = \bigcup_{n=1}^{\infty} X_n$. Suppose that each X_n has a locally convex topology so that the restriction of the topology of X_{n+1} to X_n is the given topology on X_n. Let $\mathcal{U}$ be the collection of balanced, absorbing, convex sets $\mathcal{O}$ in X for which $\mathcal{O} \cap X_n$ is open in X_n for each n. Then a) The topology generated by $\mathcal{U}$ is the strongest locally convex topology on X so that the injections $X_n \to X$ are continuous; b) The restriction of the topology on X to each X_n is the given topology on X_n; c) If each X_n is complete, so is X. The locally convex space X is called the strict inductive limit of the spaces X_n.

Q.E.D.

Define for $n \geq 1$ $G_n(\Phi') = \{\eta \in L^2(\Omega \to \Phi') : \eta[\phi] \in \mathcal{H}^{\otimes n} \ \forall \phi \in \Phi\}$. The following lemma can be proved in the same way as Theorem 5.2.2, using Lemma 5.2.2 and the fact that for each $n \geq 1$ $\mathcal{H}^{\otimes n}$ is a Hilbert subspace of $L^2(\Omega, F^W, P)$.

Lemma 5.2.4 For each $n \geq 1$, the linear space $G_n[\Phi']$ is a complete locally convex space in the topology given by the strict inductive limit of the Hilbert spaces $G_n(H_{-r})$ $r \geq 0$, i.e. $G_n(\Phi') = \varinjlim_{r\to\infty} G_n(H_{-r})$.

Multiple Wiener integral orthogonal expansions Let

$$S_Y = \{Y_n(f_n) : f_n \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-s}))\ n \geq 1,\ s \geq 0\}.$$

We shall show that S_Y is a complete set in the space $L^2(\Omega \to \Phi')$. We see from Theorem 5.2.2 and Lemma 5.2.1 that it is enough to study the completeness of the multiple Wiener integrals in each of the subspaces $L^2(\Omega \to H_{-r})$ $r \geq 0$.

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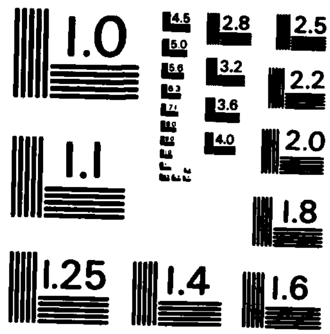

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For $r \geq 0$ let S_Y^r be the closed subspace of $L^2(\Omega \to H_{-r})$ spanned by the multiple Wiener integrals $Y_n(\cdot)$ of Proposition 5.2.3. for elements in $L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))$, i.e.

$$S_Y^r = \overline{sp}\ \{Y_n(f_n) : f_n \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))\ n \geq 1\}$$

where the closure is taken with respect to $L^2(\Omega \to H_{-r})$.

Although multiple Wiener integrals on Hilbert spaces have been studied before (Miyahara (1981)) an analog of the next result was not found in the literature.

<u>Proposition 5.2.5</u> For each $r \geq 0$

$$(5.2.16) \qquad \sigma_2(EXP(L^2(\mathbb{R}_+) \otimes H_Q), H_{-r}) \overset{\xi}{\cong} S_Y^r$$

where for $g \in \sigma_2(EXP(L^2(\mathbb{R}_+) \otimes H_Q), H_{-r})$, $g^* = (g_0^*, g_1^*, \ldots)$

$$g_n^* \in \sigma_2(H_r, (L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n}) \quad n \geq 1$$

$$\xi(g) = \sum_{n=1}^{\infty} Y_n(g_n) \qquad \text{(convergence in } L^2(\Omega \to H_{-r})).$$

<u>Proof</u> Let $g \in \sigma_2(EXP(L^2(\mathbb{R}_+) \otimes H_Q), H_{-r})$, then $g^* \in \sigma_2(H_r, EXP(L^2(\mathbb{R}_+) \otimes H_Q))$, i.e. for each $\phi \in H_s$ $g^*(\phi) \in EXP(L^2(\mathbb{R}_+) \otimes H_Q)$, $g^*(\phi) = (g_0^*(\phi), g_1^*(\phi), \ldots)$ and

$$\sum_{n=0}^{\infty} \| g_n^*(\phi) \|^2_{(L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n}} < \infty .$$

We first show that for each $n \geq 1$ $g_n^* \in \sigma_2(H_r, (L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n})$. Let $\{e_m\}_{m \geq 1}$ be a CONS in H_r, then

$$\sum_{m=1}^{\infty} \| g^*(e_m) \|^2_{EXP(L^2(\mathbb{R}_+) \otimes H_Q)} < \infty$$

and hence

$$\sum_{m=1}^{\infty} \| g^*(e_m) \|^2_{EXP(L^2(\mathbb{R}_+)\otimes H_Q)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n}} < \infty .$$

Thus for each n and $\{e_m\}_{m\geq 1}$ a CONS for H_r

$$\sum_{m=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n}} < \infty ,$$

i.e. $\quad g_n^* \in \sigma_2(H_r, (L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n}) \quad n \geq 1.$

Next, if $g \in \sigma_2(EXP(L^2(\mathbb{R}_+) \otimes H_Q), H_{-r})$ using Proposition 5.2.3 (b)

$$E\| \xi(g) \|^2_{r} = \sum_{n=1}^{\infty} E\| Y_n(g_n) \|^2_{-r}$$

$$= \sum_{n=1}^{\infty} \| \tilde{g}_n \|^2_{L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))} = \sum_{n=1}^{\infty} \| \tilde{g}_n^* \|^2_{\sigma_2(H_r, L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n})}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(\mathbb{R}_+)\otimes H_Q)^{\otimes n}} = \sum_{m=1}^{\infty} \| g^*(e_m) \|^2_{EXP(L^2(\mathbb{R}_+)\otimes H_Q)}$$

$$= \| g^* \|^2_{\sigma_2(H_r, EXP(L^2(\mathbb{R}_+)\otimes H_Q))} = \| g \|^2_{\sigma_2(EXP(L^2(\mathbb{R}_+)\otimes H_Q), H_{-r})} .$$

Then the result follows since g as above is a typical element in $\sigma_2(EXP(L^2(\mathbb{R}_+)\otimes H_Q), H_{-r}) \quad r \geq 0$.

Q.E.D.

The completeness of the multiple Wiener integrals $Y_n(f_n)$, $f_n \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))$ in $L^2(\Omega \to H_{-r})$ is now obtained.

<u>Proposition 5.2.6</u> Let $r \geq 0$ and $F \in L^2(\Omega \to H_{-r})$, $E(F) = \underline{0}$. Then

$$F = \sum_{n=1}^{\infty} Y_n(f_n) \quad \text{a.s.} \quad (\text{convergence in } L^2(\Omega \to H_{-r}))$$

where $f_n \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\otimes n}, H_{-r})) \quad n \geq 1.$

Proof By Lemma 5.2.3 $L^2(\Omega \to H_{-r}) = \sum_{n\geq 0} \oplus G_n(H_{-r})$ where for each $n \geq 1$

$$G_n(H_{-r}) = \{\eta \in L^2(\Omega \to H_{-r}) : \eta[\phi] \in \mathcal{H}^{\odot n} \quad \forall \phi \in H_r\}.$$

Then by Proposition 5.2.5 it is enough to prove that

$$\sigma_2((L^2(\mathbb{R}_+) \otimes H_Q)^{\odot n}, H_{-r}) \overset{Y_n}{\cong} G_n(H_{-r}).$$

But the last isometry follows from Proposition 5.2.4 and since from Lemma 5.1.3 and Proposition 5.1.2

$$(L^2(\mathbb{R}_+) \otimes H_Q)^{\odot n} \overset{I_n}{\cong} \mathcal{H}^{\odot n}$$

where $I_n(\cdot)$ is the real valued multiple Wiener integral of Definition 5.1.5 for elements in $L^2(\mathbb{R}_+^n \to H_Q^{\odot n})$.

Q.E.D.

The above proposition and Theorem 5.2.2 yield the next result which gives multiple Wiener integral expansions for Φ'-valued nonlinear functionals.

Theorem 5.2.3 Let $F \in L^2(\Omega \to \Phi')$, $E(F[\phi]) = 0 \quad \forall \phi \in \Phi$. Then there exists $r_F > 0$ such that $F \in H_{r_F}$ a.s. and

$$F = \sum_{n=1}^{\infty} Y_n(f_n) \quad \text{a.s.} \quad (L^2(\Omega \to H_{-r_F})\text{-convergence})$$

where $f_n \in L^2(\mathbb{R}_+^n \to \sigma_2(H_Q^{\odot n}, H_{-r_F}))$ $n \geq 1$.

Corollary 5.2.1 For $n \geq 1$ let

$$\sigma_2((L^2(\mathbb{R}) \otimes H_Q)^{\odot n}, \Phi') = \bigcup_{r=0}^{\infty} \sigma_2((L^2(\mathbb{R}) \otimes H_Q^{\odot n}), H_{-r}).$$

Then

$$G_n(\Phi') \overset{Y_n}{\cong} \sigma_2((L^2(\mathbb{R}) \otimes H_Q)^{\odot n}, \Phi').$$

The proof follows from the last part of the proof of Proposition 5.2.5 and from Lemma 5.2.4.

Stochastic integral representations for Φ'-valued nonlinear functionals

From Proposition 5.2.4 and Theorem 5.2.3 one obtains the following stochastic integral representation for elements in $L^2(\Omega \to \Phi')$. This result is the Φ'-valued analog of Theorem 6.7.1 in Kallianpur (1980), from which the idea of the proof is taken.

Theorem 5.2.4 Let $F \in L^2(\Omega \to \Phi')$. Then there exist $r_F > 0$ and a non-anticipative $\sigma_2(H_Q, H_{-r_F})$-valued process h with

$$\text{(5.2.17)} \qquad \int_0^\infty E\|h(t,\omega)\|^2_{\sigma_2(H_Q,H_{-r_F})} dt < \infty$$

such that

$$F(\omega) = \int_0^\infty h(t,\omega)dW_t \qquad \text{a.s.}$$

where the RHS in the last expression is the Φ'-valued stochastic integral of Proposition 4.2.5 with an H_{-r_F} continuous version.

Proof Since $F \in L^2(\Omega \to \Phi')$ from Lemma 5.2.1 and Theorem 5.2.3 there exists $r_F > 0$ such that $F \in H_{-r_F}$ a.s. and

$$F = \sum_{n=1}^\infty Y_n(f_n) \qquad (L^2(\Omega \to H_{-r_F})\text{-convergence})$$

where $f_n \in L^2(\mathbb{R}^n_+ \to \sigma_2(H_Q^{\otimes n}, H_{-r_F}))$ $n \geq 1$.

Next, from Proposition 5.2.4, for each $n \geq 1$

$$Y_n(f_n) = \int_0^\infty h_n(t,\omega)dW_t \qquad \text{a.s.}$$

where h_n is non-anticipative and $E\int_0^\infty \|h_n(t,\omega)\|^2_{\sigma_2(H_Q,H_{-r_F})}dt < \infty$, and the Φ'-valued stochastic integral is defined in Proposition 4.2.5.

Define $g_n = \sum_{\ell=1}^n h_\ell$, then

$$\sum_{\ell=1}^n Y_\ell(f_\ell) = \int_0^\infty g_n(t,\omega)dW_t$$

and hence

$$E\|F - \int_0^\infty g_n(t,\omega)dW_t\|^2_{-r_F} \to 0 \quad \text{as } n\to\infty . \tag{5.2.18}$$

Then using Proposition 4.2.5 (d)

$$E\int_0^\infty \|g_n(t)-g_m(t)\|^2_{\sigma_2(H_Q,H_{-r_F})}dt = E\|\int_0^\infty (g_n-g_m)(t)dW_t\|^2_{-r_F}$$
$$\to 0 \quad \text{as } n,m\to\infty$$

and hence there exists a $\sigma_2(H_Q,H_{-r_F})$-valued function g, that satisfies (5.2.17), is non-anticipative and

$$E\|\int_0^\infty h dW_t - \int_0^\infty g_n dW_t\|^2_{-r_F} \to 0 \quad \text{as } n\to\infty .$$

Then from (5.2.18) $F = \int_0^\infty h dW_t$ a.e. where h has the required properties.

Q.E.D.

Representation of Φ'-valued square integrable martingales

Definition 5.2.2 A Φ'-valued stochastic process $(X_t)_{t\in\mathbb{R}_+}$ on (Ω,F,P) is said to be a Φ'-square integrable martingale with respect to an increasing family $(F_t)_{t\in\mathbb{R}_+}$ of sub σ-fields of F if:

For each $\phi\in\Phi$ $(X_t[\phi],F_t)_{t\in\mathbb{R}_+}$ is a real valued square integrable martingale, i.e.

$$\sup_{0\le t<\infty} EX_t^2[\phi] < \infty . \tag{5.2.19}$$

Although the next proposition follows from condition (†) in Mitoma

(1981b) (page 193), we shall establish and prove it here in a way that is more convenient for use in our next theorem on the representation of Φ'-valued square integrable martingales.

Proposition 5.2.7 Let (X_t, F_t) $t \in \mathbb{R}_+$, $X_0 = 0$ be a Φ'-valued square integrable martingale. Then there exists $r_X > 0$ such that for each $t \in \mathbb{R}_+$ $X_t \in H_{-r_X}$ a.s. . Moreover, for each $\phi \in \Phi$ let $X_\infty(\phi)$ be the mean square limit of $X_t(\phi)$ as $t \to \infty$. Then there exists $\tilde{X}_\infty \in \Phi'$ a.s. $\tilde{X}_\infty \in L^2(\Omega \to \Phi')$, $\tilde{X}_\infty \in L^2(\Omega \to H_{-r_X})$ such that $\tilde{X}_\infty(\phi) = X_\infty[\phi]$ a.s. $\forall\, \phi \in \Phi$ and for $t \geq 0$

$$X_t[\phi] = E(\tilde{X}_\infty[\phi] | F_t) \quad \text{a.s.} \quad \forall\, \phi \in \Phi.$$

The proof is similar to the proof of Lemma 5.2.1 and therefore we will omit some details. In Lemma 5.2.1 we have proved that for each t, $(E(X_t[\phi])^2)^{\frac{1}{2}}$ is a lower semicontinuous function of ϕ. Then by Lemma 1, page 5, of Gelfand and Vilenkin (1964)

$$V(\phi) = \sup_{0 \leq t < \infty} (E(X_t[\phi])^2)^{\frac{1}{2}}$$

is also a lower semicontinuous convex function of ϕ. Hence, by Lemma 4.1.1, since by (5.2.19) $V(\phi) < \infty$ $\forall\, \phi \in \Phi$, there exist $\theta_X > 0$ and $s_X > 0$ such that

$$V^2(\phi) \leq \theta_X ||\phi||^2_{s_X} \quad \forall\, \phi \in \Phi.$$

Then taking $r_X \geq s_X + r_1$ one can show that

$$\text{(5.2.20)} \qquad E\Big(\sum_{j=1}^{\infty} (X_\infty(\tilde{\phi}_j))^2\Big) \leq \theta_X \theta_1 < \infty$$

where $\{\tilde{\phi}_j = (1+\lambda_j)^{-r_X} \phi_j\}_{j \geq 1}$ is a CONS for H_{r_X} with dual $\{\psi_j\}_{j \geq 1}$ which is a CONS for H_{-r_X}. Define

$$(5.2.21) \qquad \tilde{X}_\infty = \sum_{j=1}^{\infty} X_\infty(\tilde{\phi}_j)\psi_j$$

then $\tilde{X}_\infty$ is H_{-r_X}-valued a.s. and therefore $\tilde{X}_\infty \in \Phi'$ a.s. . Note that from (5.2.20) and (5.2.21)

$$E(\tilde{X}_\infty[\phi])^2 \le \theta_X \theta_1 \|\phi\|\, r_X < \infty$$

and therefore $\tilde{X}_\infty \in L^2(\Omega \to \Phi')$, $\tilde{X}_\infty \in L^2(\Omega \to H_{-r_X})$. Then it remains to prove that $X_\infty(\phi) = \tilde{X}_\infty[\phi]$ a.s. $\forall\ \phi \in \Phi$.

From (5.2.21)

$$\tilde{X}_\infty[\phi] = \sum_{j=1}^{\infty} X_\infty(\tilde{\phi}_j)\psi_j[\phi] \quad \text{a.s.} \quad \phi \in H_{r_X}$$

and then

$$\tilde{X}_\infty[\phi] = \lim_{n\to\infty} \sum_{j=1}^{n} X_\infty(\tilde{\phi}_j)\psi_j[\phi] .$$

On the other hand, since $X_\infty(\phi)$ is linear on ϕ a.s.

$$E(X_\infty(\phi) - X_\infty(\sum_{i=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{r_X}\tilde{\phi}_j))^2 = E(X_\infty(\phi - \sum_{j=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{r_X}\tilde{\phi}_j))^2$$

$$\le \theta_X \|\phi - \sum_{j=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{r_X}\tilde{\phi}_j\|^2_{r_X} \to 0 \quad \text{as } n \to \infty .$$

Then $\qquad X_\infty(\phi) = \tilde{X}_\infty[\phi]$ a.s. $\forall\ \phi \in \Phi$.

Q.E.D.

The following theorem is the Φ'-valued analog of Theorem 6.7.2 in Kallianpur (1980).

<u>Theorem 5.2.5</u> Let $(X_t, F_t^W)_{t\in\mathbb{R}_+}$, $X_0 = 0$, be a Φ'-valued square integrable martingale. Then there exists $r_X > 0$ such that X_t has an H_{-r_X} continuous version $\tilde{X}_t$ given by the stochastic integral

$$(5.2.22) \qquad \tilde{X}_t(\omega) = \int_0^t h(s,\omega)\,dW_s \quad \text{a.s.}$$

for every $t \geq 0$, where $h(t,\omega)$ is non-anticipative and

$$(5.2.23) \qquad \int_0^\infty E\| h(t,\omega) \|^2_{\sigma_2(H_Q,H_{-r_X})} dt < \infty$$

and the RHS of (5.2.22) is the Φ'-valued stochastic integral of Proposition 4.2.5.

Proof By Proposition 5.2.7 there exists $r_X > 0$ and $\tilde{X}_\infty \in \Phi'$ such that $\tilde{X}_\infty \in H_{-r_X}$ a.s. and for each $t > 0$ $X_t(\phi) = E(\tilde{X}_\infty(\phi)|F^W_t)$. Then by Theorem 5.2.3 there exists a non-anticipative $\sigma_2(H_Q,H_{-r_X})$-valued process $h(t,\omega)$ which satisfies (5.2.23) and

$$\tilde{X}_\infty(\omega) = \int_0^\infty h(\mu,\omega)dW_\mu \qquad \text{a.s. .}$$

Then the result follows using Proposition 4.2.5, from which the H_{-r_X} continuous version is also obtained.

Q.E.D.

Remarks Our results in Sections 4.2, 5.1 and 5.2 may be applied to the examples considered in Section 4.1.3 to construct stochastic integrals, multiple Wiener integrals orthogonal expansions and stochastic integral representations for nonlinear functionals of a multiparameter Gaussian process (Example 4.1.6), a cylindrical Brownian motion (Example 4.1.7) or an infinite sequence of independent Brownian motions (Example 4.1.8) as well as to the finite dimensional Gaussian process with independent increments (Example 4.1.9).

Throughout Chapters IV and V we have made the assumption that Φ is a countably Hilbert nuclear space of the kind defined in Example 4.1.1, which has the special property of having a common orthogonal set $\{\phi_j\}_{j\geq 1}$

in H_r $r \in \mathbb{R}$. Although this assumption makes some computations easier it is not essential and all our results can be obtained using only the nuclearity property of Φ.

APPENDIX

A. INFINITE TENSOR PRODUCTS OF HILBERT SPACES

We present here some material on infinite tensor products of Hilbert spaces following Guichardet (1972).

Let $(H_i)_{i\in I}$ be a family of Hilbert spaces and for each $i \in I$ a unit vector $u_i \in H_i$. Let $\underline{u} = (u_i)_{i\in I}$ be fixed. Consider a family $x_i \in H_i$ $i \in I$ such that $x_i = u_i$ *for* all but a finite number of i's. Elements of this form are called <u>Elementary Decomposable Vectors</u> and are denoted by $\underset{i\in I}{\otimes} x_i$. They form a pre-Hilbert space with inner product

$$\langle \otimes x_i, \otimes y_i \rangle = \prod_{i\in I} \langle x_i, y_i \rangle_{H_i}$$

whose completion $\overset{\underline{u}}{\underset{i\in I}{\otimes}} H_i$ is called the <u>Infinite Tensor Product Hilbert Space</u> associated with the family of unit vectors $\underline{u}$ (Guichardet (1972)). This space may also be constructed in the following manner: for each finite subset J of I construct the Hilbert tensor product

$$H_{(J)} = \overset{\underline{u}}{\underset{i\in J}{\otimes}} H_i \ .$$

For $J \subset K$, J,K finite subsets of I define a mapping

$$L_{J,K}: \underset{i\in J}{\otimes} H_i \to \underset{i\in K}{\otimes} H_i$$

by writing $\underset{i\in K}{\otimes} H_i = (\underset{i\in J}{\otimes} H_i) \otimes (\underset{i\in K-J}{\otimes} H_i)$ as

$$L_{J,K}(x) = x \otimes (\underset{i\in K-J}{\otimes} u_i) \ .$$

Then the mappings $L_{J,K}$ are isometric and form an inductive system, i.e. for $J \subset K \subset M$

$$L_{J,M} = L_{K,M} \circ L_{J,K} .$$

Then $\overset{u}{\underset{i \in I}{\otimes}} H_i$ is the inductive limit of the above system. Denote by L_J the canonical injection

$$H_{(J)} \rightarrow \overset{u}{\underset{i \in I}{\otimes}} H_i .$$

The next result identifies some elements of $\overset{u}{\underset{i \in I}{\otimes}} H_i$.

Proposition A1 Let $(x_i)_{i \in I}$, $x_i \in H_i$ be a family of vectors satisfying the following two conditions:

(1) $$\sum_i \left| \| x_i \|_{H_i} - 1 \right| < \infty$$

and

(2) $$\sum_i \left| \langle x_i, u_i \rangle_{H_i} - 1 \right| < \infty .$$

Then $\prod_i \| x_i \|_{H_i}$ exists, and it is null if and only if one of the x_i's is null. The family of vectors

$$L_J\left(\underset{i \in J}{\otimes} x_i \right)$$

has a limit in $\overset{u}{\underset{i \in I}{\otimes}} H_i$ denoted by $\otimes x_i$ whose norm is

$$\prod_{i \in I} \| x_i \|_{H_i} .$$

Moreover, $$\lim_J \left\| \underset{i \in I-J}{\otimes} x_i - \underset{i \in I-J}{\otimes} u_i \right\| = 0.$$

Proof See page 150 Guichardet (1972).

A family $(x_i)_{i\epsilon I}$ satisfying (1) and (2) in the above proposition is called a Decomposable Vector. For any two decomposable vectors we have

$$\sum_i |\langle x_i, y_i \rangle_{H_i} - 1| < \infty$$

and

$$\langle \otimes x_i, \otimes y_i \rangle = \prod \langle x_i, y_i \rangle_{H_i} .$$

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